(iv) \[ \int_0^1 f_n(x) \, dx = \int_0^{h_n} f_n(x) \, dx = 2 \int_0^{h_n} \frac{M_n}{h_n^2} \, \frac{h_n^3}{5} = \frac{2}{3} M_n h_n \]

(v) \( (f_n) \text{ bounded} \Rightarrow (M_n) \text{ is bounded} \), so \[ \int_0^1 f_n^2(x) \, dx = \int_0^{h_n} f_n^2(x) \, dx = 2 \frac{M_n^2}{h_n^2} \frac{h_n^5}{5} = \frac{2}{5} M_n^2 h_n \]

(vi)
(a) \( h_n = \frac{1}{n} \), \( M_n = \sqrt{n} \): \( M_n h_n \to 0 \) but \( M_n^2 h_n = 1 \) so \( f_n \not\to 0 \) in \( L^1 \) norm, not in \( L^2 \) norm.
(b) \( h_n = \frac{1}{n} \), \( H_n = n \): \( \int_0^1 f_n \, dx = \frac{2}{3} \text{ for all } n \), so \( f_n \) does not converge to 0 in \( L^1 \).

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f_n(x) \cos nx \, dx = 0 \text{ for all } n \text{ such that } f(x) \text{ is even} \]

(\( f(x) \) is odd, \( f(x) \) is even)

(ii) \( S_N(x) = 0 \) so \( g_n \to 0 \) uniformly on \( [0, \pi] \) (and in \( L^2 \)).

\( s_N \to f \) pointwise, but not uniformly.

\[ \int_{-\pi}^{\pi} (f_n - f_n^2) \, dx = 2 \int_{-\pi}^{\pi} \left( \frac{x^2}{h_n} \right) \, dx = 2 \frac{1}{h_n} \frac{h_n^3}{5} = \frac{2}{3} h_n \to 0 \text{ as } h_n \to 0 \]

\[ g_n \to \frac{3}{4} \text{ uniformly on } [0, \pi] \]

\[ g_n \to 0 \text{ uniformly on } [0, \pi] \]

f(x) = \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \text{ (unif conv) } a_0 = 0 \text{ sin } x = \int_{-\pi}^{\pi} f(x) \, dx = 0.

f'(x) = \sum_{n=1}^{\infty} (-n)(-n) \sin nx + (n) \cos nx.

By Parseval's equality:
\[ \frac{1}{\pi} \int_{-\pi}^{\pi} (f^2(x)) \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \left( \sum_{n=1}^{\infty} a_n^2 + b_n^2 \right) \, dx = \sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{\pi}, \]

Clearly \( \sum_{n=1}^{\infty} a_n^2 + b_n^2 \leq \sum_{n=1}^{\infty} n^2 \left( a_n^2 + b_n^2 \right) \) (equal only if \( a_n = b_n = 0 \) for \( n > 1 \)) so the claim follows.
5] Graphs of the odd extensions \( \tilde{f}(2\pi \text{- periodic}) \)

(i)

\[ \tilde{f} \text{ pw cont, } \tilde{f}' \text{ pw cont} \]

\[ s_N \to \tilde{f} \text{ in } L^2[-\pi, \pi] \]

\[ s_N(x) \to f(x) \text{ in } [0, \pi) \text{ pointwise} \]

\[ s_N(\pi) \to 0 \]

conv. not uniform in \([0, \pi]\)

(ii)

\[ f \text{ cont., } \tilde{f}' \text{ cont.} \]

\[ s_N \to f \text{ in } L^2[-\pi, \pi] \]

\[ s_N \to f \text{ uniformly in } [0, \pi] \]

(hence also pointwise)

(iii)

\[ f \text{ pw cont., } \tilde{f}' \text{ pw cont.} \]

\[ s_N \to f \text{ in } L^2[-\pi, \pi] \]

\[ s_N(x) \to f(x) \text{ in } (0, \pi) \text{ pointwise} \]

\[ s_N(0) \to 0, \ s_N(\pi) \to 0 \]

conv. not uniform in \([0, \pi]\)