INVARiance of the Laplace OPERator.

The goal of this handout is to give a coordinate-free proof of the invariance of the Laplace operator under orthogonal transformations of \( \mathbb{R}^n \) (and to explain what this means).

**Theorem.** Let \( D \subset \mathbb{R}^n \) be an open set, \( f \in C^2(D) \), and let \( U \in O(n) \) be an orthogonal linear transformation leaving \( D \) invariant (\( U(D) = D \)). Then:

\[
\Delta (f \circ U) = (\Delta f) \circ U.
\]

Before giving the proof we recall some basic mathematical facts.

**Orthogonal linear transformations.** An invertible linear transformation \( U \in L(\mathbb{R}^n) \) is **orthogonal** if \( U^*U = UU^* = I_n \) (equivalently, \( U^{-1} = U^* \)). Recall that, given \( A \in L(\mathbb{R}^n) \), \( A^* \in L(\mathbb{R}^n) \) is defined by:

\[
A^*v \cdot w = v \cdot Aw, \quad \forall v, w \in \mathbb{R}^n.
\]

Orthogonal linear transformations are **isometries** of \( \mathbb{R}^n \); they preserve the inner product:

\[
Uv \cdot Uw = v \cdot w, \quad \forall v, w \in \mathbb{R}^n, \text{ if } U \in O(n),
\]

and therefore preserve the lengths of vectors and the angle between two vectors. Orthogonal linear transformations form a **group** (inverse of orthogonal is orthogonal, composition of orthogonal is orthogonal), denoted by \( O(n) \). The matrix of an orthogonal transformation with respect to an orthonormal basis of \( \mathbb{R}^n \) is an orthogonal matrix: \( U^TU = I_n \), so that the columns of \( U \) (or the rows of \( U \)) give an orthonormal basis of \( \mathbb{R}^n \).

\[
\det(U) = \pm 1 \text{ if } U \in O(n). \quad \text{The orthogonal transformations with determinant one can be thought of as rotations of } \mathbb{R}^n. \quad \text{If } n = 2, \text{ they are exactly the rotation matrices } R_\theta:\
\]

\[
R_\theta = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}.
\]

If \( n = 3 \), one is always an eigenvalue, with a one-dimensional eigenspace (the **axis** of rotation); in the (two-dimensional) orthogonal complement of this eigenspace, \( U \) acts as rotation by \( \theta \).
Change of variables in multiple integrals. Let $A \in L(\mathbb{R}^n)$ be an invertible linear transformation. Then if $D \subset \mathbb{R}^n$ and $f : A(D) \rightarrow \mathbb{R}$ is integrable, then so is $f \circ A$ and:

$$\int_{A(D)} fdV = \int_D (f \circ A)|\text{det}A|dV.$$  

(This is a special case of the change of variables theorem.)

Multivariable integration by parts. Let $D \subset \mathbb{R}^n$ be a bounded open set with $C^1$ boundary, $f \in C^2(D)$, $g \in C^1_0(D)$. Then:

$$\int_D (\Delta f)g dV = -\int_D \nabla f \cdot \nabla g dV.$$  

Here $C^1_0(D)$ denotes the set of $C^1$ functions in $D$ which are zero near the boundary of $D$. This follows directly from Green's first identity, which in turn is a direct consequence of the divergence theorem.

Note that given $D \subset \mathbb{R}^n$ open and $P \in D$, it is easy to find a function $\varphi \in C^1_0(D)$ so that $\varphi \geq 0$ everywhere and $\varphi(P) = 1$. For example, one could let:

$$\varphi_P(x) = \max\{0, 1 - ||P - x||^2/\epsilon^2\}.$$  

This works for $\epsilon > 0$ small enough, since $\varphi_P$ is positive only in a ball of radius $\epsilon$ centered at $P$ (which does not touch the boundary of $D$ if $\epsilon$ is small.)

This can be used to observe that if $h$ is continuous in $D$, then $h \equiv 0$ in $D$ if, and only if, for any $\varphi \in C^1_0(D)$ we have:

$$\int_D h\varphi dV = 0.$$  

Indeed if $h(P) \neq 0$ for some $P \in D$ we have (say) $h(P) > 0$, hence $h > 0$ in a sufficiently small ball $B_\epsilon$ centered at $P$ (by continuity of $h$). But then $h\varphi_P \geq 0$ everywhere in $D$ and yet (since $\varphi_P \in C^1_0(D)$) we must have $\int_D h\varphi_P = 0$, so that $h\varphi_P \equiv 0$ in $D$, contradicting the fact that it is positive on $B_\epsilon$.

Proof of Theorem. By the observation just made, it is enough to show that, for all $\varphi \in C^1_0(D)$:

$$\int_D \Delta (f \circ U)\varphi dV = \int_D [(\Delta f) \circ U]\varphi dV.$$
By multivariable integration by parts, for the left-hand side:

\[ \int_D \Delta(f \circ U) \varphi dV = - \int_D \nabla(f \circ U) \cdot \nabla \varphi dV = - \int_D [(\nabla f)U] \cdot \nabla \varphi dV, \]

where we also used the chain rule to assert that \( \nabla(f \circ U) = (\nabla f)U \). On the other hand, for the right-hand side:

\[ \int_D (\Delta f) \circ U \varphi dV = \int_D [(\Delta f)(\varphi \circ U^{-1})] \circ U dV = \int_D (\Delta f)(\varphi \circ U^{-1}) dV, \]

using the change of variables theorem and the facts that \( D \) is invariant under \( U \) and \( |\det U| = 1 \).

Again from integration by parts and the chain rule we have:

\[ \int_D (\Delta f)(\varphi \circ U^{-1}) dV = - \int_D \nabla f \cdot \nabla(\varphi \circ U^{-1}) dV = - \int_D \nabla f \cdot [(\nabla \varphi)U^t] dV, \]

where we also use the fact that \( U^{-1} = U^t \).

Thus we’ve reduced the proof to showing that:

\[ \int_D [(\nabla f)U] \cdot \nabla \varphi = \int_D \nabla f \cdot [(\nabla \varphi)U^t] dV. \]

But this follows from the (easily verified) fact that for any \( n \times n \) matrix \( A \) we have:

\( (vA) \cdot w = v \cdot (wA^t), \quad \forall v, w \in \mathbb{R}^n. \)

**Application: Laplacian of radial functions.** A function \( f : B \to \mathbb{R} \) (where \( B = \{ x \in \mathbb{R}^n; |x| \leq R \} \) is a ball of some radius) is radial if it is invariant under the orthogonal group: \( f = f \circ U \), for all \( U \in O(n) \). This implies \( f \) is a function of distance to the origin only, that is (abusing the notation):

\[ f = f(r), \quad \text{if } x = r\omega \text{ with } r = |x|, \omega \in S \text{ (the unit sphere)}. \]

The theorem implies the Laplacian of a radial function is also radial:

\[ f = f \circ U, \quad \forall U \Rightarrow \Delta f = \Delta(f \circ U) = (\Delta f) \circ U \quad \forall U, \]

or \( (\Delta f)(r\omega) = g(r) \), for some function \( g(r) \) depending on \( f \), which we now compute.
From the divergence theorem on a ball $B_R$ of radius $R$ centered at 0:

$$\int_{B_R} \Delta f dV = \int_{S_R} \frac{\partial f}{\partial n} dS,$$

or:

$$\int_{S_R} g(r)r^{n-1}d\omega dr = \int_{S} f_r(R)R^{n-1}d\omega,$$

and since both integrals in $\omega$ just give the $(n-1)$-dimensional area of $S$:

$$\int_{0}^{R} g(r)r^{n-1}dr = f_r(R)R^{n-1}.$$ Differentiating in $R$ we find:

$$g(R)R^{n-1} = f_{rr}(R)R^{n-1} + (n - 1)f_r(R)R^{n-2}$$

(we assume $n \geq 2$), so finally

$$g(r) = \Delta f(r) = f_{rr} + \frac{n-1}{r}f_r.$$

**Remark.** For general (not necessarily radial) functions we have in polar coordinates $x = r\omega$:

$$\Delta f = f_{rr} + \frac{n-1}{r}f_r + \frac{1}{r^2}\Delta_S f,$$

where $\Delta_S$ is the spherical Laplacian:

$$\Delta_S f = f_{\theta\theta} \quad (n = 2), \quad \Delta_S f = f_{\phi\phi} + \frac{\cos \phi}{\sin \phi} f_\phi + \frac{1}{\sin^2 \phi} f_{\theta\theta} \quad (n = 3),$$

in polar coordinates $(r, \theta)$ (resp. spherical coordinates $(r, \phi, \theta)$, $\phi \in [0, \pi], \theta \in [0, 2\pi]$). Note the analogy between $\Delta_S$ for $n=3$ and the Laplacian for $n = 2$:

$$\Delta f = f_{rr} + \frac{1}{r}f_r + \frac{1}{r^2}\Delta_S f \quad (n = 2).$$

Making the substitutions:

$$r \rightarrow \sin \phi; \quad \frac{1}{r} = (\log r)_r \rightarrow \frac{\cos \phi}{\sin \phi} = (\log \sin \phi)_\phi, \quad \Delta_S f \rightarrow f_{\theta\theta},$$

we obtain $\Delta_S f$ for $n=3$. 

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More generally, the spherical Laplacian $\Delta_{S^n}$ on the unite sphere $S^n \subset \mathbb{R}^{n+1}$ can be described inductively in terms of the spherical Laplacian on the unit sphere $S^{n-1} \subset \mathbb{R}^n$. To do this, write $\omega \in S^n$ in spherical coordinates: as a vector in $\mathbb{R}^{n+1}$,

$$\omega = (\sin \phi, (\cos \phi) \theta) \in \mathbb{R} \times \mathbb{R}^n, \quad \theta \in S^{n-1}, \phi \in [0, \pi].$$

(This represents $S^{n-1}$ as the equator $\phi = 0$ in $S^n$.) In these coordinates, we have, for a function $f = f(\phi, \theta)$ defined on $S^n$:

$$\Delta_{S^n} f = f_{\phi \phi} + (n-1) \frac{\cos \phi}{\sin \phi} f_{\phi} + \frac{1}{\sin^2 \phi} \Delta_{S^{n-1}} f.$$

Here the operator $\Delta_{S^{n-1}} f$ involves only differentiation in the variables $\theta \in S^{n-1}$. 