Qualitative analysis of autonomous 2D systems.

1. Phase-plane analysis. Analyze the following systems by linearization at the critical points, including the items below.

SYSTEMS
1. \( x' = x + x^2, \ y' = x + y^2 \).
2. \( x' = x + y - y^2, \ y' = -x + y - 2y^2 \)
3. \( x' = x - x^2 - 2xy, \ y' = y - y^2 - 2xy \)
4. \( x' = x(1 - y), \ y' = y(1 - x) \)

OUTLINE:
(a) Find the critical points, and classify the linearized system at each critical point.
(b) If there are saddles, compute the stable/unstable eigenspaces at each saddle (these will be tangent to the stable/unstable separatrices.)
(c) Find the \( \alpha \) and \( \omega \) limits of each saddle separatrix (when they exist.)
(d) Identify other invariant sets: basins of attractors and sources, other open sets, possibly the coordinate axes.
(e) Identify finite (or half-finite) intervals of existence, when possible
(f) Include a MATLAB plot (with saddle separatrices and basins of attractors/sources highlighted) including at least two typical trajectories for each possible asymptotic behavior as \( t \to \pm \infty \).

EXAMPLE (seen in class)
\( x' = x(3 - x - y), \ y' = y(x - y - 1) \).

(a) Critical points: \( O=(0,0) \) (saddle); \( P=(3,0) \) (saddle), \( Q=(0,-1) \) (unstable node), \( R=(2,1) \) (stable spiral)
(b) \( E^s(O) = \{c(0,1)\}, \ E^u(O) = \{c(1,0)\}, \ E^s(P) = \{c(1,0)\}, \ E^u(P) = \{c(3,-5)\} \)
(c) One arc of \( W^s(O) \) has \( \alpha \)-limit, the other diverges; one arc of \( W^u(O) \) has \( P \) as \( \omega \)-limit, the other diverges.
   One arc of \( W^s(P) \) has \( O \) as \( \alpha \)-limit, the other diverges; one arc of \( W^u(P) \) has \( R \) as \( \omega \)-limit, the other diverges (it is the boundary of the basin of \( Q \).
   The coordinate axes are invariant, since \( x = 0 \) implies \( x' = 0 \) and \( y = 0 \) implies \( y' = 0 \).
   The open first quadrant is the basin of attraction \( W^s(R) \) of the sink (attractor) \( R \); the region in the open lower half-plane bounded by one arc of \( W^u(P) \) is the basin \( W^u(Q) \) of the source (repellor) \( Q \).
(e) Solutions with IC in $W^u(Q)$ or in the unstable separatrices of saddles are defined for all negative time, solutions with IC in $W^s(R)$ or the stable separatrices of saddles are defined for all positive time. In particular, one arc of $W^u(P)$ corresponds to a solution defined for all $t \in \mathbb{R}$.

2. Gradient systems.
   For each of the following functions $F(x,y)$, sketch the phase portrait (along the lines of problem 1, with the help of Matlab) for the gradient system $v' = -\text{grad} \, F(v)$.
   (a) $F(x,y) = y \sin x$.
   (b) $F(x,y) = x^2 - y^2 - 2x + 4y + 5$

3. Lotka-Volterra competition. Sketch the phase portraits (along the lines of problem 1, with the help of Matlab) for three numerical examples of Lotka-Volterra systems, one in each of the three cases:
   (i) weak competition: $\lambda_1 < \frac{k_1}{k_2} < \frac{1}{\lambda_2}$.
   (ii) strong competition: $\lambda_1 > \frac{k_1}{k_2} > \frac{1}{\lambda_2}$.
   (iii) intermediate: $\lambda_1 < \frac{k_1}{k_2} < \lambda_2$

4. Lienard’s theorem.
   Show that the second-order scalar equation $x'' + (x^6 - x^2)x' + x = 0$, $x = x(t)$, admits a non-constant periodic solution.

THEORY QUESTIONS. (Not due as homework.)
I: A gradient system cannot have non-constant periodic solutions.
II: If an autonomous system in the plane admits a conserved quantity $E(x,y)$, and $E$ is not constant on any open subset of the plane, then the system cannot have a limit cycle.