1. The Laplacian and Green’s identities

The Laplacian is the second-order differential operator defined on functions \( f \in C^2 \) by:

\[ \Delta f = \text{div}(\nabla f), \]

the divergence of the gradient vector field. In standard euclidean coordinates \((x_1, \ldots, x_N)\), it is the trace of the Hessian of \( f \):

\[ \Delta f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_N^2}. \]

In polar coordinates \( f = f(r, \omega) \), where \( r \) is distance to 0 in \( \mathbb{R}^N \) and \( \omega \in S^{N-1} \), \( \Delta \) has the expression:

\[ \Delta f = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \Delta_S f, \]

where \( \Delta_S \) is a second-order differential operator acting only on the coordinates \( \omega \). In fact, if we write a generic point \( \omega \in S \) as \( \omega = (\theta, \varphi) \in [0, \pi] \times \mathbb{R}^1 \), where \( \varphi \in [0, \pi] \) and \( \theta \in S^{N-1} = S' \), the operator \( \Delta_S \) is given by:

\[ \Delta_S f = f_{\varphi\varphi} + \frac{N-2}{\sin \varphi} f_{\theta} + \frac{1}{\sin^2 \varphi} \Delta_{S'} f, \]

where \( \Delta_{S'} \) is a second-order differential operator acting only on the variable \( \theta \). In particular, setting for the circle \( S^1 \): \( \Delta_S f = f_{\theta\theta} \) (where \((r, \theta)\) are standard polar coordinates in \( \mathbb{R}^2 \)), this defines (inductively) \( \Delta_S \) in all dimensions. For example, for \( N = 3 \), the operator \( \Delta_S \) is given by:

\[ \Delta_S f = f_{\varphi\varphi} + \frac{\cos \varphi}{\sin \varphi} f_{\theta} + \frac{1}{\sin^2 \varphi} f_{\theta\theta}. \]

An important tool in the theory of potentials is given by Green’s identities for the Laplacian, which follow easily from the divergence theorem. Recall that if \( D \subset \mathbb{R}^N \) is a smooth bounded domain, with unit outward normal vector \( n \) at points of its boundary \( \partial D \), and if \( X \) is a smooth vector field in the closed domain \( \bar{D} = D \cup \partial D \), we have:

\[ \int_D \text{div} X \, dv = \int_{\partial D} X \cdot n \, dA, \]

where \( dv = dx_1 \ldots dx_N \) is the element of volume in \( R^N \) (area if \( N = 2 \)) and \( dA \) is the element of area on \( \partial D \) (arc length if \( N = 2 \)). In the important special case \( D = B_R, \partial D = S_R \) (the \( N \)-dimensional ball, resp. \((N-1)\)-dimensional sphere centered at the origin), we have the relation (in coordinates \((r, \varphi, \theta) \in \mathbb{R}^+ \times [0, \pi] \times S^{N-2} \), as above):

\[ dv = R^{N-1} dr \, \varphi \, d\varphi \, d\theta, \quad dA = R^{N-1} d\varphi \, d\omega, \quad d\omega = (\sin \varphi)^{N-2} d\varphi \, d\theta, \]

where \( d\varphi, d\theta \) are the elements of ‘area’ in \( S^{N-1}, S^{N-2} \) (resp.); in particular \( d\theta \) is just arc length on the unit circle if \( N = 3 \).

Specializing the divergence theorem to the case \( X = g \nabla f \), where \( f, g \) are smooth functions on \( D \), we obtain:

\[ \int_D [g \Delta f + \nabla g \cdot \nabla f] \, dv = \int_{\partial D} g \frac{\partial f}{\partial n} \, dA, \quad (G1) \]
where $\frac{\partial f}{\partial n} = \nabla f \cdot n$ is the exterior normal derivative of $f$ at the boundary. This is Green’s first identity. Interchanging $f$ and $g$ and taking the difference, we obtain Green’s second identity:

$$
\int_D [f \Delta g - g \Delta f] d\text{vol} = \int_{\partial D} [\frac{\partial g}{\partial n} - \frac{\partial f}{\partial n}] dA. \tag{G2}
$$

Setting $f = g$ in (G1) we obtain the important identity:

$$
\int_D [f \Delta f + |\nabla f|^2] d\text{vol} = \int_{\partial D} \frac{\partial f}{\partial n} dA. \tag{1.1}
$$

**Definition 1.1** A function $u : D \to \mathbb{R}^n$ is harmonic in $D$ if $\Delta u = 0$.

It is clear from (1.1) that if $u$ is harmonic in $D$ (with $D \subset \mathbb{R}^n$ bounded) and either $u = 0$ on $\partial D$ (Dirichlet boundary conditions) or $\frac{\partial u}{\partial n} = 0$ on $\partial D$ (Neumann boundary conditions), then $u$ must be constant in $D$ (and the constant is zero in the Dirichlet case).

This is a good point to introduce the main boundary value problems of potential theory. A physical motivation arises from electrostatics, where Maxwell’s equations for the electric field $E$ due to a charge distribution in space characterized by the charge density function $\rho : \mathbb{R}^3 \to \mathbb{R}$ are (in appropriate units):

$$
div \mathbf{E} = \rho, \quad curl \mathbf{E} = 0.
$$

The second equation implies the existence of a ‘potential function’ $u : \mathbb{R}^3 \to \mathbb{R}$ with the property: $E = -\nabla u$, and hence $\Delta u = -\rho$. The sign ($-$) is included so that a positive charge ‘falls’ from regions of higher potential to regions of lower potential (in particular for a point charge at the origin, $u = -1/4\pi r$ increases from $-\infty$ at the origin to 0 at infinity).

The interior Dirichlet problem for a bounded domain $D$ asks for the potential $u$ inside a perfect conductor (zero charge density), given the potential $f$ on the boundary:

$$
\Delta u = 0 \text{ in } D, \quad u = f \text{ on } \partial D \quad \text{(Dirichlet)}.
$$

The interior Neumann problem for a bounded domain $D$ asks for the potential function $u$ inside a perfect conductor $D$, given the normal component of the electric field ($E_n = -\partial u/\partial n$) at boundary points:

$$
\Delta u = 0 \text{ in } D, \quad \frac{\partial u}{\partial n} = f \text{ on } \partial D \quad \text{(Neumann)}.
$$

The ‘exterior’ Dirichlet and Neumann problems are defined similarly- one wishes to find the potential outside of $D$, assuming there are no charges in the exterior.

**Exercise.** Show that if $u_1, u_2$ are solutions of the same interior Dirichlet (resp. interior Neumann) problem for the same $f$, then $u_1 \equiv u_2$ in $D$ (resp. $u_1 \equiv u_2 + \text{const.}$ in $D$).

2. Potentials in $\mathbb{R}^N$.

In this section we consider ‘whole-space’ problems. The first observation is that, unlike the one-dimensional case (where solutions of $u_{xx} = 0$ are linear), and thus define a two-dimensional space, in $\mathbb{R}^N$ for $N \geq 2$ there is a multitude of non-linear harmonic functions. For instance, denoting by $P_d^N$ the vector space of homogeneous polynomials in $n$ variables, we may consider the subspace $H_d^N \subset P_d^N$ of homogeneous harmonic polynomials of degree $d$ in $n$ variables. We have:

$$
dim(H_d^2) = 2, \text{ basis: } \{x^2 - y^2, xy\},
$$

$$
dim(H_d^2) = 2, \text{ basis: } \{x^3 - 3x^2y, y^3 - 3xy^2\},
$$

and in general $dim(H_d^2) = 2$, with basis given by the real and imaginary parts of $z^d = (x + iy)^d$. Note that $dim(P_d^N) = 2d + 1$. 
In three variables, since a general homogeneous polynomial \( p \in \mathcal{P}_d^3 \) may be written in the form:

\[
p(x, y, z) = \sum_{i=0}^d p_i(x, y) z^i, \quad p_i \in \mathcal{P}_i^2,
\]

we have \( \text{dim}(\mathcal{P}_d^3) = \sum_{i=0}^d \text{dim}(\mathcal{P}_i^2) = 1 + 2 + \ldots + (d+1) = (d+1)(d+2)/2. \) To find the dimension of the subspace \( \mathcal{H}_d^3 \subset \mathcal{P}_d^3 \), we observe that \( \mathcal{H}_d^3 \) is the kernel of the linear map defined by the Laplacian:

\[
\Delta : \mathcal{P}_d^3 \to \mathcal{P}_{d-2}^3, \quad d \geq 2.
\]

It is not hard to show that this linear map is onto, and therefore:

\[
\text{dim}(\mathcal{H}_d^3) = \text{dim}(\text{ker} \Delta) = \text{dim}(\mathcal{P}_d^3) - \text{dim}(\mathcal{P}_{d-2}^3) = (d + 1)(d + 2)/2 - (d - 1)/2 = 2d + 1.
\]

With this information, it is not hard to find bases for the \( \mathcal{H}_d^3 \):

\[
\text{dim}(\mathcal{H}_2^3) = 5, \text{ basis: } \{x^2 - y^2, xz, yz, x^2 - z^2\}
\]

\[
\text{dim}(\mathcal{H}_3^3) = 7, \text{ basis: } \{(x^6 - y^6)z, (x^2 - z^2)y, x^3 - 3xy^2, y^3, 3x^2y, y^3 - 3xy^2, z, (y^2 - z^2)x, x^2y\}.
\]

In general, one gets enough examples for a basis of \( \mathcal{H}_d^3 \) by (i) multiplying elements of \( \mathcal{H}_{d-1}^2 \) by \( z \); (ii) permuting variables.

It is also natural to look for examples of rotationally symmetric harmonic functions in \( \mathbb{R}^N \), that is, harmonic functions depending only on distance to the origin, \( r \). A harmonic \( u = u(r) \) is a solution of the ordinary differential equation:

\[
u_{rr} + \frac{N-1}{r} f_r = 0,
\]

which has solutions:

\[
u(r) = C_1 \log r + C_2, N = 2; \quad u(r) = C_1 r^{2-N} + C_2, N \geq 3.
\]

Thus we see that, except for constants, there are no rotationally symmetric harmonic functions defined on all of \( \mathbb{R}^N \) (only one \( \mathbb{R}^N \setminus \{0\} \)).

Shifting the origin to an arbitrary \( x_0 \in \mathbb{R}^N \) and choosing particular values for the constants \( C_1, C_2 \), we obtain an important definition:

**Definition 1.2.** The *Green’s function* for \( \mathbb{R}^N \) with ‘pole’ at \( x_0 \in \mathbb{R}^n \) is:

\[
G_{x_0}(x) = \begin{cases} 
\frac{1}{(N-2)\omega_{N-2}||x - x_0||^{N-2}}, & (N \geq 3); \\
\frac{1}{2\pi} \log ||x - x_0||, & (N = 2).
\end{cases}
\]

Note that the Green’s function is positive and decays to zero at infinity for \( N \geq 3 \), but changes sign and does not decay at infinity if \( N = 2 \) (this is reflected in vastly different qualitative properties for Brownian motion when for \( N = 2 \) and \( N \geq 3 \)). When \( N = 3 \), \( G_{x_0}(x) = \frac{1}{4\pi||x - x_0||} \) has the physical interpretation ‘electric potential produced by a unit point positive charge at \( x_0 \)’. It solves the equation:

\[
\Delta G_{x_0} = -\delta_{x_0},
\]

(the ‘delta function at \( x_0 \)’, and corresponds to the electric field:

\[
E(x) = -\nabla G_{x_0}(x) = \frac{1}{4\pi||x - x_0||^2}.
\]