1. Find a simple formula for the solution of the three-dimensional heat equation $u_t - \Delta u = 0$, with initial condition:

$$f(x, y, z) = x y^2 z.$$ 

*Hint:* Look for a solution of the form $u(x, y, z, t) = x v(t) z$, where $v(t)$ solves a first-order ODE, $v(0) = y$.

2. Find the solution of the heat equation in a half space $\{(x, y, z); z > 0\}$ with Neumann boundary condition $\partial u/\partial z = 0$ when $z = 0$, and given initial condition $u(x, y, z, 0) = f(x, y, z)$. The solution should be an integral involving the heat kernel. *Hint:* Use the method of reflection.

3. Let $D \subset \mathbb{R}^3$ be the ball of radius 1 centered at 0. Consider the nonhomogeneous heat equation $u_t - \Delta u = 1$ in $D$, with $u = 0$ on the boundary $\partial D$ (all $t$) and $u \equiv 0$ in $D$ at $t = 0$.

   (i) Show the problem has a steady-state solution (i.e., independent of $t$)

   $$u(r) = \frac{1 - r^2}{6}.$$ 

   (ii) Use the maximum principle to show: $u(x, t) \leq \frac{1 - ||x||^2}{6}$ for all $t$.

   (iii) Find the largest value of $a$ and the largest value of $b$ for which the minimum principle guarantees the inequality:

   $$b(1 - e^{-at}) - \frac{||x||^2}{6} \leq u(x, t),$$

   for all $x \in D$, $t \geq 0$.

4. Show there is at most one solution to the Neumann problem for the heat equation:

$$u_t - \Delta u = h(x, t) \text{ in } D, \quad \partial u/\partial n = 0 \text{ on } \partial D, \quad u(x, 0) = 0,$$

for a bounded domain $D \subset \mathbb{R}^3$.

*Hint:* Let $w$ be the difference of two solutions. Show that $I(t) = \int_D w^2(x, t)dx$ is non-increasing in $t$.

5. Find the dimension of each of the following vector spaces:

   (i) The space of all solutions of $u'' + x^2 u = 0$, $u = u(x)$;

   (ii) The eigenspace with eigenvalue $(2\pi/L)^2$ of the operator $d^2/dt^2$ on the interval $[-L, L]$, with periodic boundary conditions;

   (iii) The eigenspace of harmonic functions in the unit disk in $\mathbb{R}^2$, with homogeneous Neumann boundary conditions;

   (iv) The eigenspace with eigenvalue $\lambda = 25\pi^2$ of $\Delta$ in the unit square $[0, 1]^2 \subset \mathbb{R}^2$, with homogeneous Dirichlet boundary conditions;

   (v) The space of all solutions of the wave equation $u_{tt} = u_{xx}$ on the real line.