This is a take-home exam: You cannot talk to anyone (except me) about anything about this exam and you can only look at our book, class notes and solutions to our HW problems posted by me or done by yourself. No other reference, including the Internet.

Due date: noon on Monday (10/27). If you cannot bring it to class or to me, a scanned/typed copy by e-mail would be OK.

1. Let $G$ be a finite group, $P \in \text{Syl}_p(G)$ and $N \overset{\text{def}}{=} N_G(P)$. Prove that if $H \leq G$ with $N \leq H$, then $N_G(H) = H$.

   [Hint: Let $M \overset{\text{def}}{=} N_G(H)$. Prove that $\text{Syl}_p(H) = \text{Syl}_p(M)$ and so $n_p(H) = n_p(M)$. Use this to deduce that $H$ and $M$ have the same order.]

2. Let $G$ act on a set $S$. Define, for $g \in G$,

   \[ \chi(g) \overset{\text{def}}{=} |\{a \in S : g \in \text{stab}_G(a)\}|. \]

   [Remember that $\text{stab}_G(a) = \{g \in G : g \cdot a = a\}$.] Prove that the number of orbits of this action [remember, the orbits form a partition of $S$, so this is the number of elements in this partition], say $n$, is given by:

   \[ n = \frac{1}{|G|} \sum_{g \in G} \chi(g). \]

   Hint for 2 on the back!
[Hint: Let for $a \in S$ and $g \in G$, let

$$
\Phi(a, g) = \begin{cases} 
1, & \text{if } g \in \text{stab}_G(a); \\
0, & \text{otherwise.}
\end{cases}
$$

Clearly,

$$
\sum_{a \in S, g \in G} \Phi(a, g) = \sum_{g \in G} \left( \sum_{a \in S} \Phi(a, g) \right) = \sum_{a \in S} \left( \sum_{g \in G} \Phi(a, g) \right).
$$

What are the inner parenthesis in the latter two sums equal to?

Also, if $G \cdot a_1, \ldots, G \cdot a_n$ are all the orbits [and so $n$ is the number of orbits, as above], for a fixed $i \in \{1, \ldots, n\}$, for how many $a \in S$ do we have that $G \cdot a = G \cdot a_i$?]