On the static spacetime of a single point charge

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Einstein-Maxwell System

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu}, \]
\[ dF = 0, \]
\[ dM = 0. \]

\( T \) is the energy-momentum-stress tensor of the electromagnetic field:

\[ T_{\mu\nu} = F_{\mu\lambda} * M_{\lambda}^\nu - g_{\mu\nu} \ell. \]
Maxwell’s Aether Law

\[ M = - \ast F \]

Einstein-Maxwell-Maxwell System (EMM)

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu}, \]

\[ dF = 0, \]

\[ d \ast F = 0. \]
Reissner-Weyl-Nordström Solution

Reissner 1916, Weyl 1917, Nordström 1918.

Unique Static Spherically Symmetric solution of EMM:

\[ g_{\mu\nu}dx^\mu dx^\nu = -X(r)dt^2 + \frac{1}{X(r)}dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2), \]

\[ X(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2}, \]

\[ F = dA, \quad A = \phi(r)dt, \quad \phi = \frac{q}{r}. \]

\[ m, q \in \mathbb{R}. \]
Jebsen-Birkhoff-Eiesland Uniqueness (Rigidity)

“Any spherically symmetric solution of Einstein’s equations must also be static.”

For vacuum-Einstein: Jebsen (1921), Birkhoff (1923).

Extension to Electrovacuum: Hoffmann (1932).

Local spherical symmetry = rotation group SO(3) acts on the manifold, with orbits that are 2-dim. and spacelike.

\[ ds^2 = -A^2(t, r)dt^2 + B^2(t, r)dr^2 + C^2(t, r)(d\theta^2 + \sin^2 \theta d\phi^2). \]

Spherically symmetric electromagnetic field: \( \mathcal{L}_{U_i} F = 0 \) for \( U_i \in so(3) \).

\[ F_{tr} = f(t, r), \quad F_{\theta\phi} = c \sin \theta, \quad c = \frac{1}{4\pi} \int_S F \] (magnetic charge)
Eiesland’s Theorem

THEOREM (Eiesland, 1925) The necessary and sufficient conditions for a locally spherically symmetric Lorentzian manifold with line element

\[ ds^2 = -A^2 dt^2 + B^2 dr^2 + C^2 (d\theta^2 + \sin^2 \theta d\phi^2), \]

where \( A, B \) and \( C \) being arbitrary functions of \( t \) and \( r \), and \( C \) not a constant, to admit a one-parameter group of isometries generated by a vectorfield \( K = k_0(t, r) \partial_t + k_1(t, r) \partial_r \) are

\[
A^2 G_{r}^t = \Psi' \partial_r C \partial_t C,
\]
\[
AB (G_r^r - G_t^t) = \Psi' C \left[ A^2 (\partial_r C)^2 + B^2 (\partial_t C)^2 \right],
\]

where \( G^\nu_{\mu} = R^\nu_{\mu} - \frac{1}{2} R \delta^\nu_{\mu} \) is the Einstein tensor and \( \Psi = \Psi(C) \) is an arbitrary function of \( C \), or a constant. If the above conditions are satisfied, then the Killing field \( K \) is, up to a constant multiple, given by

\[
k_0(t, r) = -\frac{1}{e^\Psi AB} \partial_r C, \quad k_1(t, r) = \frac{1}{e^\Psi AB} \partial_t C.
\]
For the electromagnetic energy tensor,

\[ T^t_t = T^r_r, \quad T^r_t = 0, \]

(In fact this holds REGARDLESS of which aether law one uses)

Thus the uniqueness/rigidity result holds in particular for RWN: Any spherically symmetric solution of EMM is isometric to a region in the RWN spacetime.
Problems of RWN, I

(1) EITHER $|q| > m$ in which case there is a strong timelike (naked) singularity at $r = 0$:

$$K = (R_{abcd}R^{abcd})^{1/2} \sim \frac{c}{r^4} \nearrow \infty \text{ as } r \searrow 0.$$

OR $|q| \leq m$ in which case the spacetime has a complicated causal structure:

$$X(r_{\pm}) = 0, \quad r_{\pm} = m \pm \sqrt{m^2 - q^2} \quad \text{(horizons)}.$$

$q = \text{total charge of the spacetime. } m = \text{ADM mass of the spacetime.}$

For macroscopic objects $|q| \ll m$ (subextremal case).

For charged elementary particles $\frac{|q|}{m} \sim 10^{18} - 10^{22}$ (superextremal case).
Problems of RWN, II

(2) Infinite energy vs. finite total mass in the superextremal case:

\[ E_{\text{electrostatic}} = \int_0^\infty |d\phi|^2 r^2 dr = \int_0^\infty \frac{q^2}{r^2} dr = \infty \]

\[ E = mc^2. \]

Mass Renormalization (Dirac, ADM)? QFT?

(3) Origin of mass? \( m \) and \( q \) are unrelated. \( q \to 0 \) gives Schwarzschild, not Minkowski, spacetime.

(4) Trapped null geodesics: presence of photon sphere(s).

(5) Repulsive gravity in the vicinity of the naked singularity (Bonnor).
Nonlinear Electromagnetism

Born (1933) (after Mie (1912)), Born & Infeld (1934), Hoffmann (1935), Infeld & Hoffmann (1937), Schrödinger (1942),...

Basic Idea: Take a different aether law instead of Maxwell’s.

\[ S[a; D] := \int_D L_{em}(a, da), \quad \frac{\delta S}{\delta a} \bigg|_{a=A} = 0 \implies dF = 0, \quad dM = 0. \]

\[ M = \left. \frac{\partial L_{em}}{\partial f} \right|_{a=A, f=F}, \quad L_{em}(a, f) = -\ell(x(f), y(f))\epsilon[g], \]

\[ x(f) := -\frac{1}{2} \ast (f \wedge \ast f) = \frac{1}{4} f_{\mu\nu} f^{\mu\nu}, \quad y(f) := \frac{1}{2} \ast (f \wedge f) = \frac{1}{4} f_{\mu\nu} \ast f^{\mu\nu}. \]

Dominant Energy Condition: \( \ell_x > 0, \quad \ell - x\ell_x - y\ell_y \geq 0. \)

Maxwell’s choice (aether law): \( \ell_0 = x. \)

Born-Infeld’s choice: \( \ell_\beta = \frac{1}{\beta^4} \left( \sqrt{1 + 2\beta^4 x - \beta^8 y^2} - 1 \right), \beta > 0. \)
Stationary Electromagnetic Fields

\( \mathbf{K} = \) timelike Killing field of \( g \), \( X = -g(\mathbf{K}, \mathbf{K}) > 0 \), \( \mathcal{L}_K \mathbf{F} = 0 \).

\[ \mathcal{E} := \imath_K \mathbf{F}, \quad \mathcal{B} := \imath_K \ast \mathbf{F}, \]

\[ \mathcal{O} := \imath_K \ast \mathbf{M} = \ell_x \mathcal{E} + \ell_y \mathcal{B}, \quad \mathcal{H} := -\imath_K \mathbf{M} = -\ell_y \mathcal{E} + \ell_x \mathcal{B} \]

Electromagnetic Hamiltonians:

\[ H(\mathcal{E}, \mathcal{H}) := X^{-1} \mathcal{B} \cdot \frac{\partial \ell}{\partial \mathcal{B}} - \ell = X^{-1} \mathcal{B} \cdot \mathcal{H} - \ell, \]

\[ \tilde{H}(\mathcal{O}, \mathcal{B}) := -X^{-1} \mathcal{E} \cdot \frac{\partial \ell}{\partial \mathcal{E}} + \ell = X^{-1} \mathcal{E} \cdot \mathcal{O} + \ell. \]

\[ T(\mathbf{K}, \mathbf{K}) = X \tilde{H}(\mathcal{B}, \mathcal{O}), \quad R(\mathbf{K}, \mathbf{K}) = X H(\mathcal{E}, \mathcal{H}). \]
Electromagnetic Hamiltonians

\[ \tilde{H}(\mathcal{B}, \mathcal{D}) = \tilde{h}(\mu, \varpi, \sigma) \]

\[ \mu := \frac{1}{2X}(\mathcal{D} \cdot \mathcal{D} + \mathcal{B} \cdot \mathcal{B}), \]

\[ \varpi := \frac{1}{X}\mathcal{D} \cdot \mathcal{B}, \]

\[ \sigma := \frac{1}{2X}(\mathcal{D} \cdot \mathcal{D} - \mathcal{B} \cdot \mathcal{B}), \]

\[ (\tilde{h}_\mu)^2 - (\tilde{h}_\varpi)^2 - (\tilde{h}_\sigma)^2 = 1, \quad \tilde{h}_\mu > 0. \]
Spherically Symmetric Electrostatic Solutions of EMX

Electrostatic: $\mathcal{V} = \mathfrak{H} = 0$.

Let $\zeta(\mu) := \tilde{H}(0, \xi) = \tilde{h}(\mu, 0, \mu)$ be the reduced Hamiltonian. Then for all $m, q \in \mathbb{R}$,

$$X(r) = 1 - \frac{2m}{r} + \frac{2}{r} \int_{r}^{\infty} \zeta\left(\frac{q^2}{2r'^4}\right) r'^2 \, dr', \quad \phi(r) = q \int_{r}^{\infty} \zeta'(\frac{q^2}{2r'^4}) \frac{dr'}{r'^2}.$$  

is the unique spherically symmetric electrostatic (SSE) solution of EMX with reduced Hamiltonian $\zeta$.

Hoffmann (1935), Pellicer-Torrence (1969), Demianski (1986), Breton (2002), ...

Note: Given $\ell = \ell(x, y)$ let $f(t) := -\ell(-\frac{1}{2}t^2, 0)$ and let $f_*(s) = \sup_t [st - f(t)]$. Then $\zeta(\mu) = f_*(\sqrt{2\mu})$.  

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Requirements for the Reduced Hamiltonian $\zeta$

(R1) $\lim_{\mu \to 0} \frac{\zeta(\mu)}{\mu} = 1$.

(R2) $\zeta'(\mu) > 0$ and $\zeta - \mu \zeta'(\mu) \geq 0$ for all $\mu > 0$.

(R3) $\zeta'(\mu) + 2\mu \zeta''(\mu) \geq 0$ for all $\mu > 0$.

(R4) $\int_{0}^{\infty} \mu^{-7/4} \zeta(\mu) d\mu < \infty$.

(R5) There exists positive constants $J_\zeta, K_\zeta, L_\zeta$ such that

$$J_\zeta \sqrt{\mu} - K_\zeta \leq \zeta(\mu) \leq J_\zeta \sqrt{\mu} \quad \text{and} \quad \frac{J_\zeta}{2\mu^{1/2}} - \frac{L_\zeta}{\mu} \leq \zeta'(\mu) \leq \frac{J_\zeta}{2\mu^{1/2}}.$$
Analyzing the SSE metric

Kretchman scalar:

\[ \mathcal{K}^2 = R_{abcd}R^{abcd} = X''^2 + r^{-2}X'^2 + r^{-4}(1 - X)^2. \]

Recall \( X(r) = 1 - \frac{2m(r)}{r} \) with \( m(r) := m - \int_r^\infty \zeta \left( \frac{q^2}{2r'^4} \right) r'^2 dr' \).

So \( \mathcal{K} \sim r^{-3} \) if \( m(0) \neq 0 \) (as in negative mass Schwarzschild.) For RWN, \( m(0) = -\infty \).

Let \( m = \int_0^\infty \zeta \left( \frac{q^2}{2r'^4} \right) r'^2 dr' = q^{3/2}I_\zeta \). Then \( m(0) = 0 \), and in fact \( m(r) = c_1r - c_2r^3 + \ldots \) as \( r \to 0 \) with \( c_1 > 0 \), so there is a conical singularity at \( r = 0 \) with \( \mathcal{K} \sim r^{-2} \).

Given \( \bar{m} > 0 \) let \( \zeta_\beta(\mu) := \beta^{-4}\zeta(\beta^4\mu) \) with \( \beta = q^{3/2}I_\zeta/\bar{m} \). Then

\[ m_{ADM}(\zeta_\beta) = m(\infty) = \bar{m} \]

i.e. the mass is entirely of electrical nature (no bare mass).
Requirements \((R1)-(R3)\) are incompatible with having a regular center (Bronnikov 2000).

\[
(R1)-(R3) \implies \zeta(\mu) \geq c\sqrt{\mu}
\]

No Singularity at \(r = 0\) \(\implies\) \(\zeta(\mu) \rightarrow C\) as \(\mu \rightarrow \infty\).

\[
(R4) \implies \zeta(\mu) \leq C\mu^{3/4}.
\]

Thus \textit{mildest possible} singularity is when \(\zeta(\mu) \sim c\sqrt{\mu}\). This is \((R5)\).
Mass-to-Charge Ratio as a Small Parameter

Given a Lagrangian $\ell(x, y)$ and a pair $(q, m)$ there is a scaled version of the Hamiltonian such that the corresponding SSE solution has charge $q$ and mass $m$.

Let $\epsilon = \frac{m}{|q|}$. For an electron $\epsilon \sim 10^{-18}$. Then we have

$$X(r) = \begin{cases} 
1 - \frac{2m}{r} + \frac{q^2}{r^2} & \text{as } r \to \infty \\
(1 - A\epsilon^2) + \frac{B\epsilon^6}{m^2}r^2 & \text{as } r \to 0
\end{cases}$$

and

$$\phi(r) = \begin{cases} 
\frac{q}{r} & \text{as } r \to \infty \\
\frac{3\epsilon}{2} - \frac{A\epsilon^3}{2m}r & \text{as } r \to 0
\end{cases}$$

Large distance asymptotics are the same as RWN. Moreover, for sufficiently small $\epsilon > 0$:

1. There are no horizons of any kind.
2. There is a conical singularity at $r = 0$, with deficit angle $\sim \epsilon^2$.
3. The electric field has a point defect at $r = 0$. 
Geodesics and Test-charge Trajectories

Small $\epsilon \Rightarrow$ No trapped null geodesics.

Radial (zero ang. mom.) null and timelike geodesics are incomplete in one direction (fall into singularity).

Nonradial null geodesics are deflected by the singularity $d \sim \epsilon^2$.

Timelike geodesics have bound and escape orbits.

Test-charges have bound and escape orbits.

There are no static orbits.

There is NO evidence of repulsive gravity in the neighborhood of the singularity.
Conclusions

THEOREM (T.-Z. 2010) Among all electromagnetic theories which
(a) are derivable from a Lagrangian,
(b) satisfy the dominant energy condition, and
(c) in the weak field limit coincide with classical linear electromagnetics, there exists a subclass, characterized by certain growth conditions on the Hamiltonian, with the property that

1. The corresponding unique, spherically symmetric, asymptotically flat, electrostatic spacetime metric has the mildest possible singularity at its center, namely, a conical singularity on the time axis.

2. The electric potential moreover has a finite value on the axis, and the electric field has a point defect there.

3. The total electrostatic energy is finite, and is equal to the ADM mass of the spacetime.

4. By scaling the Lagrangian, one can arrange the total mass and total charge of these spacetimes to have any chosen values.

5. For small enough mass-to-charge ratio, these spacetimes have no horizons and no trapped null geodesics.

6. An analysis of geodesics and test-charge trajectories shows that unlike the Reissner-Nordström case, the singularity at the center of these particle-spacetimes is gravitationally attractive.
Where to go from here?

1. Stationary axially symmetric spacetimes with charge and spin, and no horizon. What kind of singularity? (with M. Kiessling)

2. Static solutions corresponding to two (or more) point charges. Are there Struts? (with M. Kiessling)

3. Conformal aether laws and complete integrability of EMuBI. (with S. Beheshti)

4. Radiation probing of spacetimes with conical singularities. (with F. Planchon and J. Stalker)