Integrability, Harmonicity and the Xanthopoulos Conjecture

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We survey the application of the Inverse Scattering Mechanism to Einstein Equations in the case of stationary, axisymmetric metrics; the status of several open problems will be discussed. Xanthopoulos’ geometric definition of complete integrability is given, suggesting a connection between harmonic maps and integrable systems. New evidence for the conjecture is presented through examples from nonlinear electromagnetism and M-theory.
Xanthopoulos’ Conjecture

(1984) For systems of \( n > 1 \), partial differential equations for \( n \) unknown scalar fields of two independent variables, which can be described as a harmonic mapping of Riemannian manifolds, the following are equivalent:

1. The manifold of fields (of the harmonic mapping description of the system) admits \( 2n - 1 \) independent Killing vector fields.
2. The system can be described as a Lax pair.
Outline

- The Inverse Scattering Mechanism
- Applications of ISM to Einstein Equations
- Harmonicity = “Complete Integrability”
- A Modified Conjecture & Evidence
Integrability

Definition(s): A nonlinear PDE or system of PDE is said to be integrable or an integrable system if

1. It admits closed-form solutions
2. A Lax pair can be determined
3. Inverse Scattering Mechanism (ISM) can be successfully applied
ISM: Very Brief History

In the past four decades, Inverse Scattering Theory has had a rich and varied impact on mathematical physics, allowing for extensive treatment of numerous equations:

- **KdV** \( u_t - 6uu_x + u_{xxx} = 0 \)
- **sine-Gordon** \( u_{xx} - u_{tt} = \sin u \)
- **Painlevé Equations** \( y'' = R(y', y, t) \)
- **KP** \( (u_t + u_xu + \epsilon^2u_{xxx})_x \pm u_{yy} = 0 \)
- **NLS** \( i\psi_t + \frac{1}{2}\psi_x^2 = K|\psi|^2\psi \)
- **Einstein Vacuum**, **Einstein Maxwell**

Gardner, Greene, Kruskal, Miura, Zabusky, Lax, Faddeev, Gel’fand, Levitan, Marchenko, Gibbon, Caudrey, Bullough, Eilbeck, Zakharov, Mihailov, Shabat, Belinskii, Ablowitz, Kaup, Newell, Segur, Hirota, ...
ISM: The Scheme

- Start with the nonlinear 2D PDE $u_t = F(u, u_x, u_{xx}, \ldots)$.

- Examine the associated scattering problem for, say, the stationary 1D Schrödinger Equation having as potential the unknown $u$

  $$L\psi = \lambda\psi \quad L = -\frac{d^2}{dx^2} + u(x, t) \quad \text{(simplest case)}$$

- Assume $u(x, t)$ decays rapidly at infinity and let $u(x, 0)$ be Cauchy data at $t = 0$. The direct scattering problem consists of finding full set of scattering data $S(\lambda, 0)$ produced by potential $u(x, 0)$.

- Determine the equation of evolution for the scattering data $S(\lambda, 0) \rightarrow S(\lambda, t)$.

- Reconstruct the potential $u(x, t)$ via scattering data $S(\lambda, t)$. 
Is this possible? Yes, if two “miracles” occur...

m1: Evolution eqns for scattering data $S(\lambda, t)$ can be integrated

m2: Gelfand-Levitan-Marchenko integral equations, can be solved

Key point: In integrable cases, the eigenvalues of the associated spectral problem are independent of time and the eigenfunctions $\psi$ obey an evolution equation which is first order in the time $t$

$$\psi_t = M\psi.$$ 

Here, $M$ is a differential operator depending on $u$ and derivatives of $x$ only. It is this equation which allows us to find the exact time dependence of the scattering data. $L, M$ are called the Lax Pair for the system.
Existence of two equations for the eigenfunction $\psi$ means that a self-consistency condition must be satisfied. This condition coincides with the nonlinear PDE of interest.

What about second-order equations? Development of the ISM showed that most of the known 2D equations can be represented as self-consistency conditions for a pair of matrix equations:

$$
\psi_x = U\psi \quad \psi_t = V\psi
$$

where $\psi = (\psi_1, \psi_2)^T$ is a column matrix and $U, V$ are $2 \times 2$ matrices depending on $x, t, \lambda$.

Compatibility condition then becomes $U_t - V_x + [U, V] = 0$ for all $\lambda$. 
Example: sine-Gordon Equation

Let $\zeta = \frac{1}{2}(x + t)$ and $\eta = \frac{1}{2}(x - t)$ and consider the system

$$U = i\lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{i}{2} \begin{pmatrix} 0 & u_\zeta \\ u_\zeta & 0 \end{pmatrix} \quad V = \frac{1}{4i\lambda} \begin{pmatrix} \cos u & -\sin u \\ i\sin u & -\cos u \end{pmatrix}.$$ 

Then the compatibility condition can be computed to be

$$\begin{pmatrix} 0 & u_\zeta \eta - \sin u \\ u_\zeta \eta - \sin u & 0 \end{pmatrix} = 0 \iff u_\zeta \eta = \sin u,$$

and the spectral problem can also be written down easily

$$\psi_{1,\zeta} = i\lambda \psi_1 + \frac{i}{2} u_\zeta \psi_2 \quad \psi_{2,\zeta} = -i\lambda \psi_2 + \frac{i}{2} u_\zeta \psi_1,$$

from which the direct scattering problem can be solved. Evolution in $\eta$ is computed to then evaluate the inverse scattering transform and find $u(\zeta, \eta)$; see [Novikov, Manakov, Pitaevsky & Zakharov].
Example: Principal Chiral Model Let \((\zeta, \eta) \in \mathbb{R}^2\) and consider the action

\[
S = \int \frac{1}{2} Tr \left( g\zeta g^{-1}_\eta \right) \sqrt{\det g} \, d\zeta d\eta
\]

where \(g = g(\zeta, \eta)\) is an element of a fixed Lie group \(G \subseteq GL(n, \mathbb{C})\) and \(g^{-1}_\eta = (g^{-1})\eta\). The associated Euler equation

\[
g\zeta \eta = \frac{1}{2} \left( g\zeta g^{-1}_\eta g + g_\eta g^{-1} g\zeta - (\ln \alpha)_\eta g\zeta - (\ln \alpha)\zeta g_\eta \right),
\]

is known as equations of the principal chiral field on the group \(G\)–a free field in two-dimensional spacetime with values in \(G\).
Observe that for $A = i\alpha g_\zeta g^{-1}$ and $B = i\alpha g_\eta g^{-1}$ the Lagrangian becomes

$$L = \frac{1}{2} Tr (g_\zeta g_\eta^{-1}) = \frac{1}{2} Tr AB$$

and the field equations are equivalent to the system

$$A_\eta + B_\zeta = 0$$
$$A_\eta - B_\zeta - i [A, B] = 0.$$ 

Here, the second equation is the compatibility condition of $A$ and $B$; notice it also looks like the compatibility condition for a linear system

$$i \psi_\zeta = U \psi$$
$$i \psi_\eta = V \psi$$

for $\psi$ a column function of $\zeta, \eta$ and $U, V$ $n \times n$ matrix functions of $\zeta, \eta$ and $\lambda \in \mathbb{C}$. 
How to determine the matrices $U, V$

- Assume $U$ has $N_1$ poles: $U = U_0 + \sum_{n=1}^{N_1} \frac{U_n}{\lambda - \lambda_n}$. Similarly for $V$.

- Substitute $U, V$ into the linear system and equate all residues, obtaining $N_1 + N_2 + 1$ equations for the functions $U_n, V_m$, $1 \leq n \leq N_1$, $1 \leq m \leq N_2$.

- Use symmetries of the system to reduce the equations and solve.

As a simple example, assume $U, V$ have simple poles in $\lambda$,

$$U = U_0 + \frac{U_1}{\lambda - \lambda_1}, \quad V = V_0 + \frac{V_1}{\lambda - \mu_1}.$$  

Gauge invariance $\Rightarrow U_0 = V_0 = 0$ and Relativistic invariance $\Rightarrow -\lambda_1 = \frac{1}{\mu_1}$. Direct calculation to conclude $U_1 = A$, $V_1 = B$ and for $\lambda_1 = 1$, recover sine-Gordon system.
Remarks:

1. Dressing technique, or “vesture method” is carried out to generate eigenfunctions $\psi$ and recover solutions $g$ to the original field equations (see EVE).

2. Higher-dimensional generalisation of two-dimensional spacetime suggested by examining

\[
D_1\psi = U\psi \\
D_2\psi = V\psi 
\]

where $U, V$ are $n \times n$ matrices depending on $(x_1, \ldots, x_k)$ and $D_j$ are first order, constant coefficient linear differential operators, $D_j = \sum_{i=1}^{k} a_i^{(j)} \frac{\partial}{\partial x_i}$, $j = 1, 2$. Such an operator foreshadows and motivates the operators required in the Einstein vacuum and Einstein-Maxwell settings.
ISM and the Einstein (Vacuum) Equations

Start with Vacuum Equations

\[ R_{\mu\nu} = 0, \quad \mu, \nu = 0, \ldots, 3 \]

General plan: Assume stationary axisymmetric metric of the form

\[ ds^2 = f(t, z)(dz^2 - dt^2) + g_{ab}(t, z)dx^a dx^b \quad a, b = 1, 2 \]

where \((x_0, x_1, x_2, x_3) = (t, x_1, x_2, z)\) and \(z = \zeta + \eta, \; t = \zeta - \eta\).

Reduced system has \(f(\zeta, \eta)\) and \(g_{ab}(\zeta, \eta)\) as unknowns. Rewrite as a linear eigenvalue problem by generalising \(\partial_\zeta\) and \(\partial_\eta\). Apply dressing technique to solve a system of \(n\) algebraic equations.
Our motivation is the Principal Chiral Model. Set \( \det g = \alpha^2 \) and define

\[
A = -\alpha g^\zeta g^{-1} \quad B = \alpha g^\eta g^{-1}.
\]

EVE can be rewritten as

\[
\begin{align*}
(\ln f)^\zeta (\ln \alpha)^\zeta & = (\ln \alpha)^\zeta \zeta + \frac{1}{4\alpha^2} Tr A^2 \\
(\ln f)^\eta (\ln \alpha)^\eta & = (\ln \alpha)^\eta \eta + \frac{1}{4\alpha^2} Tr B^2 \\
(\ln f)^\zeta \eta & = \frac{1}{4\alpha^2} Tr AB - (\ln \alpha)^\zeta \eta \\
A \eta - B \zeta & = 0.
\end{align*}
\]

Since \( f \) can be found by quadrature once \( g \) is given, the field equations reduce to the fourth equality and an integrability condition on \( A \) and \( B \) \( (g^\zeta \eta = g^\eta \zeta) \).
Task is to find a linear system \((U, V)\) for which

\[
A\eta - B\zeta = 0
\]

\[
A\eta + B\zeta + \frac{1}{\alpha} [A, B] - \frac{\alpha\eta}{\alpha} A - \frac{\alpha\zeta}{\alpha} B = 0,
\]

appear as compatibility condition. Key idea is to generalise \(\partial\zeta = \partial_1, \partial\eta = \partial_2\) to \(D_1, D_2\) with \(D_j = \partial_j - p_j(\zeta, \eta, \lambda) \partial\lambda\) and to choose \(U, V\) to contain \(\lambda\) as simple poles

\[
D_1\psi = U\psi = \frac{A}{\lambda - \alpha} \psi
\]

\[
D_2\psi = V\psi = \frac{B}{\lambda + \alpha} \psi
\]
Determine $p_j$ by computing $[D_1, D_2] = 0$:

$$0 = \frac{\lambda - \alpha}{\lambda + \alpha} \left( p_1 - \frac{2\lambda \alpha \zeta}{\lambda - \alpha} \right) B - \frac{\lambda + \alpha}{\lambda - \alpha} \left( p_2 + \frac{2\lambda \alpha \eta}{\lambda + \alpha} \right) A.$$

Coefficients of polynomial in $\lambda$ vanish for choices of $p_j$ below

$$D_1 = \partial_{\zeta} - \frac{2\lambda \alpha \zeta}{\lambda - \alpha} \partial_{\lambda} \quad D_2 = \partial_{\eta} + \frac{2\lambda \alpha \eta}{\lambda + \alpha} \partial_{\lambda}.$$ 

Computing $[D_1, D_2]$ directly recovers wave equation (again)

$$[D_1, D_2] = \alpha \zeta \eta \frac{4\lambda^2}{\lambda^2 - \alpha^2} \partial_{\lambda} = 0 \iff \alpha \zeta \eta = 0.$$
Gauge Invariance: The generalised system having compatibility condition \( D_2 U - D_1 V + [U, V] = 0 \) possesses gauge invariance.

\[
\tilde{U} = fU f^{-1} + f_\zeta f^{-1}, \quad \tilde{V} = fV f^{-1} + f_\eta f^{-1}.
\]

Then both the linear system and its integrability condition are satisfied for \( U \to \tilde{U}, \ V \to \tilde{V} \) and \( \tilde{\psi} \to f \psi \). That is, if \( D_1 \psi = U \psi, \ D_2 \psi = V \psi \) and \( U, V \) compatible, then \( D_1 \tilde{\psi} = \tilde{U} \tilde{\psi} \) and \( D_2 \tilde{\psi} = \tilde{V} \tilde{\psi} \) for \( f = f(\zeta, \eta) \) nonsingular matrix not depending on \( \lambda \).

Relativistic Invariance: The system is also relativistically invariant. Consider the coordinate transformation \( \hat{\zeta} = \gamma \zeta, \hat{\eta} = \gamma^{-1} \eta \) and set

\[
\hat{U} = \gamma^{-1} U, \quad \hat{V} = \gamma V.
\]

Then \( D_1 \psi = U \psi \) and \( D_2 \psi = V \psi \Rightarrow \hat{D}_1 \psi = \hat{U} \psi \) and \( \hat{D}_2 \psi = \hat{V} \psi \), where \( \hat{D}_1 = \partial_{\hat{\zeta}} - \frac{2\lambda \alpha \hat{\zeta}}{\lambda - \alpha} \partial_\lambda \) and \( \hat{D}_2 = \partial_{\hat{\eta}} + \frac{2\lambda \alpha \hat{\eta}}{\lambda + \alpha} \partial_\lambda \). Visibly, the compatibility condition holds as well.
Recapitulation

The goal is to solve the vacuum equations, equivalently to find \( g \) by way of \( \psi, A, B \). The procedure of finding new solutions thus depends of choosing an initial “seed” \( g_0 \), which in turn gives \( A_0, B_0 \). One then solves the linear system in \( \psi \) to obtain a generating matrix \( \psi_0(\zeta, \eta, \lambda) \). Setting \( \psi = \chi \psi_0 \), new solutions are constructed by solving the linear system of \( \chi \). Conditions placed on the dressing matrix \( \chi \) reduce the candidates to those which will isolate physically relevant matrices \( g \), (e.g. \( g \) real, symmetric). Cauchy-Kovalevskaya Theorem ensures existence of solutions near \( \lambda = 0 \) since the solution is recovered in terms of the eigenfunction \( \psi(\zeta, \eta, \lambda = 0) = g(\zeta, \eta) \).
The Dressing Matrix

Let $g_0$ be a particular solution of the Einstein Equations $(f_0, g_0)$, forming the matrices $A_0, B_0$, and consider the problem

$$D_1 \psi = \frac{A_0}{\lambda - \alpha_0} \psi, \quad D_2 \psi = \frac{B_0}{\lambda + \alpha_0} \psi$$

with $\alpha_0^2 = \det g_0$. Note that $(\alpha_0)_\eta \zeta = 0$. Given a seed solution $\psi_0(\zeta, \eta, \lambda)$ to the linear system, construct the function

$$\psi(\zeta, \eta, \lambda) = \chi(\zeta, \eta, \lambda) \psi_0(\zeta, \eta, \lambda).$$

If $\psi$ satisfies $D_1 \psi = \frac{A}{\chi - \alpha_0} \psi$ for some $A = A(\zeta, \eta)$ (similarly for $D_2, B$), then $\chi$ must satisfy the following relations

$$D_1 \chi = \frac{1}{\lambda - \alpha_0} (A \chi - \chi A_0), \quad D_2 \chi = \frac{1}{\lambda + \alpha_0} (B \chi - \chi B_0).$$
Conditions are placed on $\chi$ by way of the solution $g$

$$g \text{ real } \Rightarrow \tilde{\chi}(\bar{\lambda}) = \chi(\lambda)$$

$$g \text{ symmetric } \Rightarrow g = \chi(\lambda)g_0 \tilde{\chi}(\alpha^2 / \lambda)$$

Assume $\chi$ has simple poles in $\lambda$

$$\chi = I + \sum_{k=1}^{N} \frac{R_k}{\lambda - \mu_k},$$

where $R_k$ are matrices and $\mu_k$ are constants, neither of which depend on $\lambda$. Reduce the linear system to a set of algebraic equations on the coefficients $R_k$ by applying the symmetry conditions.

If $\chi$ has $N$ simple poles at $\lambda = \mu_k$ then $\chi$ has $N$ poles at $\lambda = \bar{\mu}_k$; further, $\chi^{-1}$ has $N$ poles at $\nu_k = \frac{\alpha^2}{\mu_k}$, $N$ poles at $\overline{\nu_k}$.
Assume all poles occur in complex pairs and substitute this into the linear system for \( \chi \). Computing residues, \( \mu_k \) must satisfy a system of first order equations

\[
\mu_{k,\zeta} = \frac{-2 \alpha \zeta \mu_k}{\mu_k - \alpha} \quad \mu_{k,\eta} = \frac{2 \alpha \eta \mu_k}{\mu_k + \alpha}.
\]

Think of these as the evolution equations for the spectral parameter.

Deduce \( R_k \) is degenerate so that \((R_k)_{ab} = n_a^{(k)} m_b^{(k)} = \vec{n} \vec{m}^t \), where \( \vec{n} = n_a \) and \( \vec{m} = m_b \) are column vectors, \( a, b = 1, 2 \) (suppress \((k)\) index). Similarly, \( \chi^{-1}(\mu_k) = \vec{q} \vec{p}^t \) and hence

\[
0 = \left[ \vec{m}_\zeta^t + \vec{m}^t \frac{A_0}{\mu_k - \alpha} \right] \vec{q} \quad 0 = \left[ \vec{m}_\eta^t + \vec{m}^t \frac{B_0}{\mu_k + \alpha} \right] \vec{q}.
\]
Goal is to specify $R_k$ (equivalently, $\vec{m}, \vec{n}$) given initial solution $\psi_0$. Fix $\lambda = \mu_k$ in $\psi_0^{-1}(\zeta, \eta, \lambda) := M$ and differentiate $I = \psi_0 \psi_0^{-1}$ with respect to $\zeta$ to find matrix equation isolating $\vec{m}$

$$0 = M^{(k)}_\zeta + M^{(k)} \frac{A_0}{\mu_k - \alpha}.$$ 

Use symmetry $\lambda \leftrightarrow \alpha^2/\lambda$ and above equation to compute residue at $\lambda = \alpha^2/\mu_k$, we obtain a system of $N$ algebraic relations on the matrices $R_k$. 
Introduce an electromagnetic field into the stress energy tensor in EE for \((M^4, g)\):

\[
R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = F_\mu^\lambda - \frac{1}{4}g_{\mu\nu}F^\alpha_\beta F_{\alpha\beta}
\]

\[
D^\alpha F_{\alpha\beta} = 0, \quad D^\alpha * F_{\alpha\beta} = 0
\]

Assume stationarity and axisymmetry (again), but this time, in terms of complex Ernst potentials, \(\epsilon(\rho, z)\) and \(\Phi(\rho, z)\):

\[
(\epsilon + \bar{\epsilon} + 2\Phi\bar{\Phi})\nabla^2 \epsilon = 2(\nabla \epsilon + 2\Phi\nabla \Phi) \cdot \nabla \epsilon
\]

\[
(\epsilon + \bar{\epsilon} + 2\Phi\bar{\Phi})\nabla^2 \Phi = 2(\nabla \epsilon + 2\Phi\nabla \Phi) \cdot \nabla \Phi
\]
Advantages

- Can be regarded as equations determining harmonic mapping $f : M \rightarrow M'$ where

  \[
  M : \quad ds^2 = d\rho^2 + dz^2 + \rho^2 d\phi \\
  M' : \quad ds'^2 = F^{-2}|d\epsilon + 2\Phi d\Phi|^2 - 4F^{-1}d\Phi d\bar{\Phi}
  \]

  for $F = \Re \epsilon + \Phi \bar{\Phi}$.

- **Definition:** Let $(M, g_{mn})$ and $(N, g_{AB})$ be Riemannian manifolds and $f : M \rightarrow N$, $f = \{f^A\}$ be a smooth mapping from $M$ to $N$, expressed in terms of the $n = \dim N$ scalar fields $\{f^A\}$ on $M$. The mapping $f$ is called harmonic if and only if

  \[
  g^{mn}D_m D_n f^A + \Gamma^A_{BC}(D_m f^B)(D_n f^C)g^{mn} = 0
  \]

  where $D_m$ is the covariant derivative of $(M, g_{mn})$ and $\Gamma^A_{BC}$ are the Christoffel symbols of $(N, g_{AB})$. 
• $M'$ is a Riemannian symmetric space with isometry group $SU(2, 1)$. There are 8 Killing vectors $Z_k, k = 1 \ldots 8$ generating the isometry group, having associated Killing 1-forms $\tau^k$, as well as corresponding conserved currents

$$j^\mu_k = Z^a_k p^\mu_a = Z^a_k g^{\mu\nu} \partial_\nu f^b g'_{ab} = f^* \circ \tau^k, \quad k = 1 \ldots 8 \text{ (Noether)}.$$

• Since $f$ is a harmonic map and $M'$ is a symmetric space, the conservation laws (in conjunction with the Maurer-Cartan Equations $d\omega + \omega \wedge \omega = 0$) are pulled back to precisely the EM field equations

$$d \ast \Sigma = 0$$
$$d\Sigma + \Sigma \wedge \Sigma = 0$$

for $\Sigma = f^* \circ \tau^k$
• Using $3 \times 3$ matrix representation of generators $X_{\mu}$ of $SU(2, 1)$, define a connection 1-form $W = X_\mu \tau^\mu = -(dP)P^{-1}$ for a Hermitian matrix $P$ (det = 1) with entries in terms of $\epsilon, \bar{\epsilon}, \Phi, \bar{\Phi}$. $P$ satisfies a compact form of the field equations

$$d[(\ast dP)P^{-1}] = 0.$$ 

Here, $\ast$ denotes the Hodge star.

• It is in this form which Inverse Scattering can be applied!
Coordinate-free ISM for Einstein-Maxwell Linear eigenvalue problem for \( \Psi = \Psi(\lambda, E) \) is

\[
D\Psi = -\Omega \Psi
\]

for \( D = d - \left( \frac{\partial \theta}{\partial \lambda} \right)^{-1} d\theta \frac{\partial}{\partial \lambda} \), and \( \Omega = aW + b\alpha ^* W, \alpha ^2 = \det g \). Here, \( a, b \) are complex functions defined on \( \mathbb{R}^2 \times \mathbb{C} \) whose limits approach 1 and 0, respectively, as \( \lambda \to 0 \).

After imposing a symmetric space property on \( P \) (quadratic constraint), explicit construction of dressing matrix \( \chi \) can be accomplished as in vacuum case.
Xanthopoulos’ Conjecture

Definition: A system of partial differential equations which can be described by a harmonic mapping is said to be completely integrable when the corresponding \((n\text{-dimensional})\) manifold of fields admits \(2n - 1\) linearly independent Killing vector fields.

Conjecture For systems of \(n, n > 1\), partial differential equations for \(n\) unknown scalar fields of two independent variables, which can be described as a harmonic mapping of Riemannian manifolds condition (i) implies condition (ii):

(i) The manifold of fields (of the harmonic mapping description of the system) admits \(2n - 1\) independent Killing vector fields.

(ii) The system can be described as a Lax pair.
1. **SAS Vacuum Equations**: We have constructed the Lax Pair by way of ISM. Further, a harmonic map formulation exists for two-dimensional constant scalar curvature hyperboloid as target manifold, having metric $ds^2 = 1x^2 (dx^2 + dy^2)$. This manifold has 3 Killing fields.

2. **SAS Einstein-Maxwell**: We have given a harmonic map formulation, ISM applies and there are 8 Killing fields.

3. **Two-Dimensional nonlinear $\sigma$-models**: Zakharov & Mikhailov establish harmonicity with manifold of fields being unit hyperboloids of constant curvature, maximally symmetric spaces having $n(n+1)/2$ Killing fields.

4. **SU(n) Self-Dual Source-Free Yang-Mills**: ISM applies $2(n^2 - 1)$ Killing fields.
Special Cases

For $n = 2, 3$ one can show that if space has $2n-1$ Killing fields, then it is maximally symmetric. Case $n=2$ is trivial, since $2n-1 = n(n+1)/2$. Case $n=3$ uses fact that an $n$-dimensional manifold cannot have $n(n+1)/2-1$ Killing fields without having, in fact $n(n+1)/2$ of them ($n \neq 4$). Hence a three-dimensional manifold having $2(3)-1=5$ Killing fields must have 6 and thus be maximally symmetric.

In both cases, this implies $N$ is constant scalar curvature, so $N \sim \mathbb{E}^n$, $S^n$, $\mathbb{R}P^n$, or $n$-dimensional simply connected hyperbolic space (Cartesian coords, or nonlinear $\sigma$-models).
Questions and Other Possible Directions

⋄ ISM applies when the base manifold is effectively two-dimensional and the manifold of fields (target manifold) is a Riemannian symmetric space. Even in case of Vacuum equations, $\phi$ coordinate was supressed. Complete integrability “doesn’t see” the dimensionality of manifold of coordinates $M$.

⋄ Preliminary evidence in M-Theory: $D = 4, \mathcal{N} = 1$ supergravity description of five-dimensional heterotic $M$-theory has a Lagrangian description of a harmonic map

$$S_4 = \beta \int d^4 x \sqrt{-g} \left( \frac{1}{2} R + K_{i,j} \partial_\mu \Phi^i \partial^{\mu} \bar{\Phi}^j \right)$$

with Kähler metric $K_{i,j}$, and an expected 10 Killing fields. Applicability of ISM is currently under investigation.
Conjecture Condition (i) implies condition (ii):

(i) The manifold of fields (of the harmonic mapping description of the system) admits \(2n - 1\) independent Killing vector fields.

(ii) The system can be described as a Lax pair.

Stronger Conjecture: (i) and (ii) are, in fact, equivalent.

Even Stronger Conjecture: (i) is equivalent to (ii‘):

(ii‘) The system is integrable by way of Inverse Scattering.
Thank you very much!