A compactness theorem for the Yamabe problem on manifolds with boundary

Marcelo Disconzi - joint work with Marcus Khuri
The Yamabe problem

· Given \((M, g)\) (compact without boundary), find \(\tilde{g} \in [g]\) such that \(R_{\tilde{g}} = \text{constant}\).

· Equivalent to finding solution of

\[
L_g u + K u^{\frac{n+2}{n-2}} = 0, \quad u > 0
\]  

(1)

where \(L_g = \Delta_g - c(n) R_g\) = conformal Laplacian,
\(c(n) = \frac{n-2}{4(n-1)}\), \(K = \text{constant}\).

· \(u\) solution of (1) \(\Rightarrow \tilde{g} = u^{\frac{4}{n-2}} g\) has \(R_{\tilde{g}} = c(n)^{-1} K\).

· Equation (1) is \textit{conformally invariant}. 
Introduction...

- Analytical point of view: rich source of problems.
- Standard calculus of variation techniques fail to apply (critical exponent).
- Yamabe problem was solved in the affirmative (Yamabe, Trudinger, Aubin, Schoen).
“Counting” solutions

Manifolds \((M, g)\) fall into 3 different classes according to the \textit{Yamabe invariant} \(Y(M)\):

\[
Y(M) = \inf_{\tilde{g} \in [g]} \frac{\int_M R_{\tilde{g}} dV_{\tilde{g}}}{\left(\int_M dV_{\tilde{g}}\right)^{n-2}}
\]

\[
\cdot \ Y(M) = 0 \Rightarrow \exists \tilde{g} \in [g] \bigg| R_{\tilde{g}} = 0 \quad (K = 0).
\]

\[
\cdot \ Y(M) > 0 \Rightarrow \exists \tilde{g} \in [g] \bigg| R_{\tilde{g}} = +1 \quad (K > 0).
\]

\[
\cdot \ Y(M) < 0 \Rightarrow \exists \tilde{g} \in [g] \bigg| R_{\tilde{g}} = -1 \quad (K < 0).
\]
Non-uniqueness

For the case $Y(M) > 0$, solutions to

$$L_g u + K u^{\frac{n+2}{n-2}} = 0, \quad u > 0$$

are not unique, and the set of solutions is “big”.
Main question

What can be said about the set $\Phi$ of solutions to:

$$L_g u + K u^{\frac{n+2}{n-2}} = 0, \quad u > 0$$

in the case $Y(M) > 0$ ($K > 0$)?
Describing solutions when $Y(M) > 0$

Set of solutions $\Phi$ is:

- Non-compact when $(M, g) \approx (S^n, g_0)$.

If $(M, g) \not\approx (S^n, g_0)$, $\Phi$ is compact (in the $C^2$ topology) when:

- $M$ is locally conformally flat (Schoen);
- $\dim(M) = 3$ (Schoen & Zhang);
- $\dim(M) \leq 5$ (Druet);
- $\dim(M) \leq 7$ (Marques);
- $\dim(M) \leq 11$ (Li & Zhang);
- Conjecture: $\Phi$ is compact in $C^2$ if $(M, g) \not\approx (S^n, g_0)$. 
Compactness of $\Phi$?

- Conjecture is false. Counter-examples: Brendle $n \geq 52$, Brendle & Marques $n \geq 25$.

- But: compactness is true in all remaining cases ($n \leq 24$) (Schoen, Khuri & Marques).
Yamabe problem on manifolds with boundary.

· Given \((M, g)\) with \(\partial M \neq \emptyset\), find \(\tilde{g} \in [g]\) such that \(R_{\tilde{g}} = \text{constant}\) and \(\kappa_{\tilde{g}} = \text{constant}\).

· Equivalent to finding positive solution of

\[
\begin{align*}
L_g u + Ku^{\frac{n+2}{n-2}} &= 0, & \text{in } M, \\
\partial_{\nu_g} u + \frac{n-2}{2} \kappa_g u &= \frac{n-2}{2} cu^{\frac{n}{n-2}}, & \text{on } \partial M,
\end{align*}
\]

\(u\) solution of (2) \(\Rightarrow \tilde{g} = u^{\frac{4}{n-2}} g\) has \(R_{\tilde{g}} = c(n)^{-1}K\) and \(\kappa_{\tilde{g}} = c\).

· Several authors (Escobar, Marques, Brendle, Chen, Almaraz, Ahmedou): most cases solved.

· Problem (2) is \textit{conformally invariant}.
Yamabe problem on manifolds with boundary.

\[
\begin{aligned}
L_g u + K u \frac{n+2}{n-2} &= 0, \quad u > 0, \quad \text{in } M, \\
\partial_{\nu_g} u + \frac{n-2}{2} \kappa_g u &= \frac{n-2}{2} cu^{\frac{n}{n-2}}, \quad \text{on } \partial M,
\end{aligned}
\]

· If \( K \neq 0 \) and \( c \neq 0 \) \( \Rightarrow \) equation \textit{and} the boundary condition are nonlinear.

· Assume then that either \( K \) or \( c \) is zero.

· Geometrically deforming the manifold to one with:
  · constant nonzero scalar curvature and zero mean curvature on the boundary \( (K \neq 0, c = 0) \);
  · zero scalar curvature and constant nonzero mean curvature on the boundary \( (K = 0, c \neq 0) \).

· Here we focus on the first of these two cases.
Compactness?

Similarly to the case to the case $\partial M = \emptyset$, if $(M, g) \not\cong (S^n_+, g_0) \Rightarrow$ question of compactness. Compactness was proven:

- $K < 0$ and $c = 0$ (Han & Li);
- $K > 0$, $c \in \mathbb{R}$, $M$ locally conformally flat and $\partial M$ umbilic (Han & Li);
- $K = 0$, $c > 0$, $M$ locally conformally flat and $\partial M$ umbilic (Felli & Ahmedou);
- $K = 0$, $c \in \mathbb{R}$, $n \geq 7 +$ a generic condition on the trace-free part of $\kappa_{ij}$ (Almaraz).
- Except for the locally conformally flat case, nothing known so far when $K > 0$ (equation non-linear).
Compactness

· Consider:

\[
\begin{cases}
L_g u + K u^p = 0, \quad u > 0, & \text{in } M, \\
\partial_{\nu_g} u + \frac{n-2}{2} \kappa_g u = 0, & \text{on } \partial M,
\end{cases}
\]

where \( p \in (1, \frac{n+2}{n-2}] \) (including subcritical approximations) and \( K > 0 \).

Define:

\[
\Phi_p = \left\{ u > 0 \mid u \text{ solves } 3 \right\},
\]
Main theorem (Compactness)

Theorem 1. (Disconzi and Khuri) Let \((M^n, g)\) be a smooth compact Riemannian manifold of dimension \(3 \leq n \leq 24\) with umbilic boundary, and which is not conformally equivalent to the standard hemisphere \((S^n_+, g_0)\). Then for any \(\varepsilon > 0\) there exists a constant \(C' > 0\) depending only on \(g\) and \(\varepsilon\) such that

\[
C^{-1} \leq u \leq C \quad \text{and} \quad \| u \|_{C^{2,\alpha}(M)} \leq C
\]

for all \(u \in \bigcup_{1+\varepsilon \leq p \leq \frac{n+2}{n-2} \Phi_p} \), where \(0 < \alpha < 1\).
Remarks

· Theorem expected to be false when \( n \geq 25 \) (analogously to the boundaryless case).

· Umbilicity tensor

\[
T_{ij} := \kappa_{ij} - \frac{1}{n-1} \kappa g_{ij}
\]

is conformally invariant.

· \( \partial M \) umbilic: \( T_{ij} \equiv 0 \); conformally invariant condition (\( \sim \) totally geodesic).

· Compactness provides an alternative proof of the solution to the Yamabe problem \( \Rightarrow \) refined existence theorems (Leray-Schauder degree of solutions).
Ingredients of the proof

- Elliptic theory: enough to obtain a priori bound
  \[ \| u \|_{C^0(M)} \leq C \] for all \( u \in \Phi_p \).

- Assume the contrary: sequences \( \{u_i\} \in \Phi_p \)
  \( \{x_i\} \subset M \), \( x_i \to \bar{x} \), \( u_i(x_i) \to \infty \).

- Conformal invariance: \( g_i \mapsto \tilde{g}_i = \) “good properties”
  (use \( T_{pq} \equiv 0 \)).

- \( u_i \approx G_i(x_i, \cdot) = \) Green’s function for \( L_{g_i} \) centered at \( x_i \).

- \( G_i(x_i, x) \approx |x - x_i|^{2-n} + A \).

- Positive Mass Theorem: \( A > 0 \).

- \( \lim \inf_{r \to 0} P_r(u_i) \geq 0 \) and \( \lim \inf_{r \to 0} P_r(u_i) = -A \).
Behavior of conformally invariant quantities

- Whenever “blow-up” $u_i(x_i) \to \infty$ occurs, it is expected that conformally invariant quantities will vanish to high order.

- Here we assume $T_{pq} \equiv 0$.

- So we focus on the Weyl tensor $W_{pq}$. 
Weyl vanishing Theorem

Theorem 2. Let $g$ be a smooth Riemannian metric defined in the unit $n$-half ball $B_1^+$, $6 \leq n \leq 24$. Suppose $B_1^+ \cap \mathbb{R}^{n-1}$ is umbilic and that $\exists$ sequence of positive solutions $\{u_i\}$ to

$$\begin{cases} L_g u_i + K u_i^{p_i} = 0, & u_i > 0 \quad \text{in } B_1^+ \\ \partial_{\nu_g} u_i + \frac{n-2}{2} \kappa_g u_i = 0, & \text{on } B_1^+ \cap \mathbb{R}^{n-1} \end{cases}$$

$p_i \in (1, \frac{n+2}{n-2}]$, such that for any $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that $\sup_{B_1^+ \setminus B_{\varepsilon}^+} u_i \leq C(\varepsilon)$ and $\lim_{i \to \infty} (\sup_{B_1^+} u_i) = \infty$. Then the Weyl tensor $W_g$ satisfies, for some integer $\ell > \frac{n-6}{2}$,

$$|W_g|(x) \leq C|x|^\ell$$
Setting and notation

· \( \{ g_i \}_{i=1}^{\infty} \) sequence of metrics, \( g_i \stackrel{C^k(M)}{\rightarrow} g, \ k = k(n) \).

· \( \{ u_i \} \) sequence of solutions to (where \( K = n(n-2) \)):

\[
\begin{cases}
  L_{g_i} u_i + Ku_i^{\frac{n+2}{n-2}} = 0, \ u_i > 0, \ & \text{in} \ M \\
  \partial_{\nu_{g_i}} u_i + \frac{n-2}{2}\kappa_{g_i} u_i = 0, \ & \text{on} \ \partial M
\end{cases}
\]

· \( g = e^h, \ h : B_{\rho}(p) \rightarrow n \times n \) symmetric matrices.

· \( \partial^\alpha h(p) = \) curvature terms.

· In conformal normal coordinates at \( p \) (Lee & Parker, Escobar):

· \( \det(\tilde{g}_i)(x) = 1 + O(|x|^N), \ N \) as large as we want.

\( \Rightarrow d\ vol_g = dx \) and other good properties.
Blow up points

- Recall we assume: $x_i \to \bar{x}$, $u_i(x_i) \to \infty$
- We can assume: $\bar{x} \in \partial M$ (boundaryless case otherwise; proof is local).
- Assume for simplicity: $\{x_i\} \subset \partial M$.
- Geometry constrains behavior of $u_i$ near $x_i$, assume:
  - $x_i$ local max of $u_i$
  - $u_i$ has profile:

$(x_i$ is isolated simple blow-up point).
Behavior of $u_i$

· Put $\varepsilon_i = u_i(0)^{-\frac{2}{n-2}}$ (rate of blow up).
· We obtain an approximation $u_i \approx u_{\varepsilon_i} =$ “standard bubble”:

$$u_{\varepsilon_i}(x) = \varepsilon_i^{\frac{n-2}{2}} (\varepsilon_i^2 + |x|^2)^{\frac{2-n}{2}}$$

Notice $u_i \approx |x|^{2-n}$.
· $u_i \approx u_{\varepsilon_i}$ is not enough; need a correction term:

$$u_i \approx u_{\varepsilon_i} + z_{\varepsilon_i}$$

· $z_{\varepsilon_i}$ solves an equation related to the linearization of $L_g u + K u^{\frac{n+2}{n-2}} = 0$. 
Boundary condition for $z_\varepsilon$?

- Construction of $z_\varepsilon$ uses equation, but not the boundary condition for $u$.

- Need to compute $\partial_n z_\varepsilon |_{x^n=0}$.

- $z_\varepsilon$ given explicitly in terms of polynomial expressions in the coordinates whose coefficients are derivatives of $h$:

$$H_{ij}^{(k)}(x) = \sum_{|\alpha|=k} \frac{1}{\alpha!} \partial^\alpha h_{ij}(0) x^\alpha$$

- From the explicit construction “enough” to compute $\partial_n^\ell h(0)$. 
Estimate

Take conformal normal coordinates at \(x_0 \in \partial M\) and choose a large integer \(N\). Then there exists a constant \(C\), depending only on \(N\) such that:

\[
\sum_{|\alpha|=2}^{N} |\kappa,\alpha| \varepsilon^{|\alpha|} \leq C \sum_{|\alpha|=2}^{N} \sum_{i,j=1}^{n-1} |T_{ij,\alpha}| \varepsilon^{|\alpha|}
\]

where \(\alpha\) denotes partial derivatives in the variables \(x^1, \ldots, x^{n-1}\) evaluated at the origin.

Umbilicity \(\Rightarrow \kappa \sim O(|x'|^N), \ x = (x', x^n)\).
Estimate...

Also:

\[
\sum_{|\alpha|=2}^{N} \sum_{j=1}^{n-1} |h_{nj,\alpha}| \varepsilon^{|\alpha|} \leq C \sum_{|\alpha|=1}^{N-1} \sum_{i,j=1}^{n-1} |T_{ij,\alpha}| \varepsilon^{|\alpha|+1}
\]

\[
\sum_{|\alpha|=1}^{N} \sum_{i,j=1}^{N-1} |\partial_n h_{ij,\alpha}| \varepsilon^{|\alpha|} \leq C \sum_{|\alpha|=1}^{N} \sum_{i,j=1}^{n-1} |T_{ij,\alpha}| \varepsilon^{|\alpha|}
\]

\[
\sum_{|\alpha|=1}^{N} |\partial_n h_{nn,\alpha}| \varepsilon^{|\alpha|} \leq C \sum_{|\alpha|=1}^{N} \sum_{i,j=1}^{n-1} |T_{ij,\alpha}| \varepsilon^{|\alpha|}
\]

Umbilicity $\Rightarrow h_{nj}, \partial_n h_{ij}, \partial_n h_{nn} \sim O(|x'|^N)$,

$i, j = 1, \ldots, n - 1, x = (x', x^n)$. 
Boundary conformal normal coordinates

· Start with conformal normal coordinates at $x_0 \in \partial M$.

· Goal: make a further conformal change such that $\kappa_{\tilde{g}} = 0$, preserving properties of conformal normal coordinates.

· If $\tilde{g} = e^{2f} g$ then $\kappa_{\tilde{g}} = e^{-f} (\kappa_g + \frac{\partial f}{\partial \nu_g})$.

· “Solve” $\frac{\partial f}{\partial \nu_g} + \kappa_g = 0$.

· Estimate: $\kappa_g = O(|x'|^N) \Rightarrow f = O(|x|^N) \Rightarrow \det(\tilde{g}) = 1 + O(|x|^N)$. 
Higher order estimates

* Our main estimate:

\[ h_{nj}, \partial_n h_{ij}, \partial_n h_{nn} \sim O(|x'|^N) \]

\((i, j \leq n - 1)\) combined with boundary conformal normal coordinates yields: for \(\partial_n h, \partial^2_n h\), e.g.:

\[ \partial^2_n h_{nj}(x', 0) = O(|x'|^N), \quad j = 1, \ldots, n - 1 \]
Boundary conditions

Take boundary conformal normal coordinates $x_i \in \partial M$:

- **umbilicity** $\Rightarrow \kappa_{ij} = 0 \Rightarrow \partial M = \{x^n = 0\}$.

- $\partial_v u + \frac{n-2}{2} \kappa_g u = 0 \mapsto \partial_v u = 0$.

- Higher order estimates $\Rightarrow \partial_n z_\varepsilon |_{x^n=0} = 0$

- Direct computation: $\partial_n u_\varepsilon |_{x^n=0} = 0$

- Estimates on $h$ then imply:

\[ \partial_v (u - u_\varepsilon - z_\varepsilon) \approx 0 \]
Approximation \( u \approx u_\varepsilon + z_\varepsilon \)

\[
|\partial^m (u - u_\varepsilon_i - z_\varepsilon_i)(x)| \leq C\varepsilon_i \frac{[\frac{n-2}{2}] - 1}{2} \sum_{k=2}^{\frac{n-2}{2}} |H^{(k)}|^2(x_i)(\varepsilon_i + |x|)^{2k + 2 - n - m}
\]

where: \(|H^{(k)}|^2 = \sum_{kl} \sum_{|\alpha| = k} \frac{1}{\alpha!} |\partial^\alpha h_{kl}(0)|^2.\)

\[\cdot\ u_i \mapsto u_\varepsilon_i + z_\varepsilon_i \text{ up to an error quadratic in } h \text{ and its derivatives evaluated at the origin.}\]

\[\cdot\ \text{Denote all such quadratic errors by } Q_i(\partial^\alpha h).\]

\[\cdot\ \text{Proof uses representation formula, hence our need for } \partial_\nu (u - u_\varepsilon - z_\varepsilon) \approx 0.\]
Pohozaev identity

\[ P(r, w) = \int_{\partial B_r^+} \left( \left( \frac{n-2}{2} w + x^k \partial_k w \right) \frac{\partial w}{\partial \nu_0} - \frac{1}{2} x^k \nu_0^k |\nabla_0 w|^2 \right) ds \]

Multiplying \( L_{g_i} u_i + Ku_i^{\frac{n+2}{n-2}} = 0 \) by \( x^k \partial_k u_i + \frac{n-2}{2} u_i \) and integrating by parts:

\[ o(1)Q_i (\partial^\alpha h) + \text{error}(\varepsilon_i) + P(r, u_i) \geq \]

\[ - \int_{B_r^+} \left( \frac{1}{2} x^k \partial_k R_{g_i} + R_{g_i} \right) (u_{\varepsilon_i}^2 + 2u_{\varepsilon_i} z_{\varepsilon_i}) dx + B.I. \]

where B.I. = boundary integrals.

\[ \cdot \text{ Boundary integrals are handled with our estimates } h = O(|x'|^N). \]
Key estimate

The integral \( J = - \int_{B^+} \left( \frac{1}{2} x^k \partial_k R + R \right) \left( u^2 + 2 u z \right) \)
can be approximated by a quadratic form on Taylor polynomials of \( h \):

\[
J \approx I(\partial^\alpha h, \partial^\beta h)
\]

**Proposition 3.** The quadratic form \( I \) is positive definite for \( n \leq 24 \).
**Estimate on** $Q(\partial^\alpha h)$...

It follows:

$$o(1)Q_i(\partial^\alpha h) + \text{error}(\varepsilon_i) \geq - \int_{B^+_r} \left( \frac{1}{2} x^k \partial_k R_{g_i} + R_{g_i} \right) (u_{\varepsilon_i}^2 + 2u_{\varepsilon_i}z_{\varepsilon_i}) \, dx$$

$$\approx I_i(\partial^\alpha h, \partial^\alpha h)$$

$$\geq Q_i(\partial^\alpha h) > 0$$

Therefore:

$$\text{error}(\varepsilon_i) \geq Q_i(\partial^\alpha h) > 0$$

and $Q_i(\partial^\alpha h) \to 0$ as $i \to \infty$. 
Consequences

Vanishing of $\partial^\alpha h(\bar{x})$:

\[ Q_i(\partial^\alpha h) \to 0 \Rightarrow \partial^\alpha h(\bar{x}) = 0. \]

Weyl vanishing:

\[ \partial^\beta W(x_i) \sim \partial^\alpha h. \]

\[ |\nabla^\ell g_{W_g}(\bar{x})|^2 = 0, \quad 0 \leq \ell \leq \left[ \frac{n-6}{2} \right]. \]

Expansion of the metric $g$ around $\bar{x}$:

\[ \text{Weyl vanishing } \Rightarrow \quad g = \delta + O(|x|^{\left[ \frac{n-2}{2} \right]+1}). \]
Blow up of $M$

- Green’s function:
  - Asymptotic expansion:
    \[ G(\bar{x}, x) \sim |x - \bar{x}|^{2-n} + A \]
  - Boundary condition:
    \[ \partial_{\nu} G + \frac{n - 2}{2} \kappa_{g} G = 0 \]
- Blow-up of the manifold: $\hat{g} = G^{\frac{4}{n-2}} g$.
- $(M \setminus \{\bar{x}\}, \hat{g})$ is scalar flat and has totally geodesic $\partial M$. 
Application of the Positive Mass Theorem

· Doubling of \((M \setminus \{x\}, \hat{g})\) and:

\[
g = \delta + O(|x|^{[\frac{n-2}{2}]+1})
\]

⇒ well defined mass.

· Positive Mass Theorem: \(A > 0\).

· Calculation:

\[
\lim \inf_{r \to 0} P(r, G) = -A < 0
\]

where as before:

\[
P(r, w) = \int_{\partial B_{r}^{+}} \left( \left( \frac{n-2}{2} w + x^k \partial_k w \right) \frac{\partial w}{\partial \nu_0} - \frac{1}{2} x^k \nu_0^k |\nabla_0 w|^2 \right) ds
\]
**Alternative estimate of** \( \liminf_{r \to 0} P(r, G) \)

- Recall from Pohozaev identity:

\[
o(1)Q_i(\partial^{\alpha} h) + \text{error}(\varepsilon_i) + P(r, u_i) \geq \]

\[
- \int_{B_r^+} \left( \frac{1}{2} x^k \partial_k R_{gi} + R_{gi} \right) (u_{\varepsilon_i}^2 + 2u_{\varepsilon_i}z_{\varepsilon_i}) \, dx + B.I.
\]

- Previous estimates + Weyl vanishing + key estimate:

\[
P(r, u_i) \geq o(1)
\]

- limit in \( i \): \( u_i \to G(\bar{x}, \cdot) \Rightarrow P(r, u_i) \to P(r, G) \).

- limit in \( r \) \( \Rightarrow \liminf_{r \to 0} P(r, G) \geq 0 \).
Positivity of $I(\partial^\alpha h, \partial^\beta h)$.

Boundaryless case:

- Properties of homogeneous polynomials.
- Theory of harmonic polynomial ($L^2$ decomposition, spherical harmonics, etc).
- Integration by parts.
- After several estimates and computations, the problem reduces to verify that certain matrices are positive definite.
- That is done with help of Maple.
Positivity of $I(\partial^\alpha, \partial^\beta)$.

For the case with boundary:

· Need to verify that the integration by parts argument works, with the extra boundary integrals (along $\partial' B_+$) being an error.

· All such integrals will involve quantities either appearing in the main estimate ($h_{ni}, \partial_n h_{ij}, \partial_n h_{nn}, i, j \leq n - 1$) or involve $\partial_n z_\varepsilon(x', 0): O(|x'|^N)$.

· Argument goes through.