Quasi-local mass, static vacuum metrics, and fill-ins of nonnegative scalar curvature

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Quasi-local mass

Consider a spacetime:

\[ (M^3, g) \]

... with spacelike slice \((M, g)\) that is totally geodesic
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- Scalar curvature \( R \) of \( g \) is observed energy density \( \geq 0 \)
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How much mass is contained in bounded region \( \Omega \)?

- Scalar curvature \( R \) of \( g \) is observed energy density \((\geq 0)\)
- \( \Omega \) may contain horizons (outermost minimal surfaces)
Bartnik data

- “Quasi-local mass” of $\Omega$ ought to depend only on geometry near $\partial \Omega$: in particular the Bartnik data $(\Sigma, \gamma, H)$
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We assume:
- $\Sigma \cong S^2$
- $\gamma$ has positive Gauss curvature: $K_{\gamma} > 0$
- $H > 0$
Examples of quasi-local mass

1. Hawking mass:

\[ m_H(\Sigma, \gamma, H) = \sqrt{\frac{\text{area}_\gamma(\Sigma)}{16\pi}} \left( 1 - \frac{1}{16\pi} \int_{\Sigma} H^2 dA_\gamma \right) \]
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2. Brown–York mass:

\[ m_{BY}(\Sigma, \gamma, H) = \frac{1}{8\pi} \int_{\Sigma} (H_0 - H) dA, \]

where \( H_0 \) is mean curvature of embedding \((\Sigma, \gamma) \hookrightarrow \mathbb{R}^3\).
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3. Bartnik mass:

\[ m_B(\Sigma, \gamma, H) = \inf\{ m_{ADM}(M, g) \}, \]

where \((M, g)\) is an asymptotically flat extension of \((\Sigma, \gamma, H)\) with \( R \geq 0 \), no horizons.
Test case 1: round sphere in $\mathbb{R}^3$
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Euclidean space

- $m_H(\Sigma) = 0$
- $m_{BY}(\Sigma) = 0$
- $m_B(\Sigma) = 0$
Schwarzschild metric, $m > 0$

$$g_{ij} = \left(1 + \frac{m}{2r}\right)^4 \delta_{ij}$$

apparent horizon of black hole
Test case 2

Schwarzschild metric, $m > 0$

$$g_{ij} = \left(1 + \frac{m}{2r}\right)^4 \delta_{ij}$$

- $m_H(\Sigma) = m$
- $m_{BY}(\Sigma) > m$
- $m_B(\Sigma) = m$
Test case 3

Schwarzschild metric, $m > 0$

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$(\Sigma, \gamma, H)$
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Fill-ins

Motivation:

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Key idea: (non-)existence of fill-ins

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(fill-in without black holes)
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Key idea: (non-)existence of fill-ins
- Does given data \((\Sigma, \gamma, H)\) have a fill-in?

(fill-in without black holes)\[ (\Omega^3, g) \quad \overset{R \geq 0}{\longrightarrow} \]

(fill-in with black holes)\[ (\Sigma^2, \gamma, H) \quad \overset{R \geq 0}{\longrightarrow} \]
Trichotomy

Bartnik data \((\Sigma, \gamma, H)\) falls into one of three classes:

- **negative type**: no fill-in exists
- **zero type**: a fill-in exists, but only without black holes
- **positive type**: a fill-in exists, with black holes

**Examples**

- **negative type**: concentric sphere in Schwarzschild (uses positive mass theorem)
- **zero type**: off-center sphere in Schwarzschild (uses Riemannian Penrose inequality)
- **positive type**: concentric sphere in Schwarzschild
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Examples

- **negative type**: concentric sphere in \(m < 0\) Schwarzschild
  - uses positive mass theorem with corners (Miao)
- **zero type**: off-center sphere in \(m > 0\) Schwarzschild
  - uses Riemannian Penrose inequality (Bray)
- **positive type**: concentric sphere in \(m > 0\) Schwarzschild
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Characterization of type zero

Proposition

Every fill-in \((\Omega, g)\) of type zero data is \textbf{static vacuum}, meaning

\[ \text{Ric}(-u(x)^2 \, dt^2 + g) = 0 \]

for some function \(u > 0\) on \(\Omega\). (Implies \(g\) has zero scalar curvature.)
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Proof

- Take some fill-in \((\Omega, g)\), assume **not** static vacuum.
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Proof

- Take some fill-in $(\Omega, g)$, assume not static vacuum.
- Corvino: locally increase scalar curvature away from boundary.
- Introduce a small black hole in region with positive scalar curvature.
- Conclude data is of positive type.
Shi–Tam Theorem

Theorem (Shi–Tam, 2002)

If \((\Sigma, \gamma, H)\) has a fill-in, then

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\int_{\Sigma} (H_0 - H) \, dA_{\gamma} \geq 0,
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with equality if and only if data comes from Euclidean \(\mathbb{R}^3\).
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Remarks

• Nonnegativity of Brown–York mass
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- Nonnegativity of Brown–York mass
- Consequence: for \(H\) “too large”, \((\Sigma, \gamma, H)\) has no fill-in.
- Fundamentally depends on positive mass theorem
Main Theorem

Idea of Bray: consider data $(\Sigma, \gamma, \lambda H)$ as function of parameter $\lambda \in \mathbb{R}^+$. 

Theorem (J.)

There exists a unique $\lambda_0 > 0$ such that $(\Sigma, \gamma, \lambda H)$ is positive type if and only if $\lambda \in (0, \lambda_0)$. Moreover, data is negative type if $\lambda > \lambda_0$.

Remarks

• Conjecture: data is zero type at $\lambda = \lambda_0$.

• $\lambda_0$ is canonically associated to $(\Sigma, \gamma, \lambda H)$. 
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Outline of proof

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\[ I_+ = \{ \lambda > 0 : (\Sigma, \gamma, \lambda H) \text{ is positive type} \} \]
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R = -2\dot{H} + 2K - H^2 - \|k\|^2
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- **Step 4:** Rule out latter: if \( \lambda_0 \) corresponded to a positive type fill-in, could bump up mean curvature slightly.
Application to quasi-local mass

- Use $\lambda_0$ to define a quasi-local mass of Bartnik data $(\Sigma, \gamma, H)$.

- On radius $r$ coordinate sphere in Schwarzschild metric, $\lambda_0 = 1 + m^2 r^{-1} - m^2 r > 1$.

- New definition. In general:
  $$m(\Sigma, \gamma, H) = \frac{\text{area} \\gamma(\Sigma)}{16\pi} \left(1 - \frac{1}{\lambda_0^2}\right)$$

- Recovers "mass coordinate sphere in Schwarzschild."
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• Recovers “$m$” on coordinates spheres in Schwarzschild.
Test cases, revisited

- $m(\Sigma) = 0$
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General properties

For this definition of quasi local mass:

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- \( m(\Sigma) \geq 0 \) if \( \Sigma \) has a fill-in. Equality implies any fill-in is static vacuum.
- Black hole limit property.
- Monotonicity in spherical symmetry.
Further questions

- lift the restriction $K_\gamma > 0$
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- monotonicity in general?
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- lift the restriction $K_\gamma > 0$
- monotonicity in general?
- handle general slices of spacetimes