Examples Jordan Normal Form

UTK – M531 – Ordinary Differential Equations I
Fall 2004, Jochen Denzler, TR 11:10–12:25, Ayres 309B

(These examples were generated with the help of symbolic algebra software, and also the
calculations were done using such software.)
diag(λ₁, . . . , λₙ) denotes the diagonal matrix (size n×n) with the diagonal entries as specified.

I may write column vectors like
\[
\begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix}
\]
as transposed rows [1, 2, 3]ᵀ for typographical reasons.

Example 1:

The matrix
\[
A = \begin{bmatrix}
\frac{44}{3} & -\frac{82}{3} & -20 \\
\frac{19}{3} & -\frac{35}{3} & -10 \\
4 & -8 & -4
\end{bmatrix}
\]
has characteristic polynomial
\[
\det(A - \lambda I) = -\lambda^3 - \lambda^2 + 10\lambda - 8 = -(\lambda - 2)(\lambda - 1)(\lambda + 4)
\]
where the factorization is based on eyeballing (guesswork) to find \(\lambda = 1\) as one root, and
then the quadratic formula.

Since all eigenvalues are distinct, we can diagonalize the matrix \(A = SDS^{-1}\) with \(D = \text{diag}(1,2,-4)\). (Any other order like eg. \(D = \text{diag}(1,-4,2)\) would do just as well, with a
different \(S \neq S\) (similarly permuted. So we have chosen the numbering \(λ_1 = 1, λ_2 = 2, λ_3 = -4\). Let’s find corresponding eigenvectors \(v_1, v_2, v_3\):

\(v_1\) is calculated as a solution to the linear system \((A - \lambda_1)v_1 = 0\):
\[
\begin{align*}
\frac{41}{3}v_1^{(1)} + \frac{82}{3}v_1^{(2)} + (-20)v_1^{(3)} &= 0 \\
\frac{19}{3}v_1^{(1)} + \frac{-38}{3}v_1^{(2)} + (-10)v_1^{(3)} &= 0 \\
4v_1^{(1)} + (-8)v_1^{(2)} + (-5)v_1^{(3)} &= 0
\end{align*}
\]
(Upper indices denote components of the vector, lower indices identify vectors.) Gauss
elimination from this system produces: \(v_1^{(1)} = 2v_1^{(2)}\), \(v_1^{(3)} = 0\). We may therefore choose
\(v_1 = [2, 1, 0]^T\). Any nonzero multiple would have been just as legitimate a choice (and
would have led to a somewhat different matrix \(S\)).

Likewise we obtain \(v_2 = [1, -1, 2]^T\) and \(v_3 = [4, 2, 1]^T\), where again nonzero multiples
would have been just as legitimate choices. We have therefore found (our choice of) the
matrix \(S\):
\[
S = \begin{bmatrix}
2 & 1 & 4 \\
1 & -1 & 2 \\
0 & 2 & 1
\end{bmatrix}
\]
(Vertical lines inserted to make eigenvectors more clearly visible). It is another bunch of
linear equations to find its inverse (if you need it):

\[ S^{-1} = \frac{1}{3} \begin{bmatrix} 5 & 7 & -6 \\ 1 & -2 & 0 \\ -2 & 4 & 3 \end{bmatrix} \]

You can now check explicitly that \( AS = SD \), or \( A = SDS^{-1} \).

**Example 2:**

This example must be higher dimensional, because its purpose is to display several bells and whistles at the same time. Took me quite a while to come up with an example with somewhat decent numbers. Let


The characteristic polynomial is

\[ \det(A - \lambda I) = -\lambda^7 - 10\lambda^6 - 41\lambda^5 - 90\lambda^4 - 120\lambda^3 - 112\lambda^2 - 80\lambda - 32 = -(\lambda + 2)^5(\lambda^2 + 1) \]

where the factorization is a conspiratively designed lucky ‘coincidence’. So \( \lambda_1 = -2 \) is an eigenvalue with algebraic multiplicity 5 and \( \lambda_{6,7} = \pm i \) are single eigenvalues. To find the geometric multiplicity of \( \lambda_1 \), we must actually solve the linear system \((A + 2I)v = 0\). Let me do this ‘by hand’, with ‘unsystematic’ row transformations (i.e., smart pivoting) to preserve nice numbers:

\[
\begin{align*}
90 & 6 & -2 & 2 & -36 & -45 & -27 & -3 & 6 & -2 & 2 & 4 & 0 & 1 \\
93 & 0 & 0 & 0 & -40 & -45 & -28 & 93 & 0 & 0 & 0 & -40 & -45 & -28 \\
93 & 0 & 0 & 0 & -40 & -45 & -28 & \text{step1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-560 & -12 & 3 & -3 & 237 & 275 & 169 & -560 & -12 & 3 & -3 & 237 & 275 & 169 \\
-554 & 4 & -1 & 1 & 243 & 269 & 168 & 6 & 16 & -4 & 4 & 6 & -6 & -1 \\
\end{align*}
\]

**step 1:** subtract 3rd row from 2nd and 4th; subtract 5th row from 6th and add three times to 7th.

**step 2:** add 5×last row to 5th, then subtract 3rd row from last, then use 2nd row to produce leading 0’s in 1st, 3rd, 6th row

**step 3:** move 4th row to bottom, 2nd row to top, and use it to create more zeros in the 1st column

\[
\begin{align*}
0 & 64 & -21 & 21 & -15 & 2 & -3 & 6 & -2 & 2 & 4 & 0 & 1 \\
-3 & 6 & -2 & 2 & 4 & 0 & 1 & 0 & 64 & -21 & 21 & 31 & -15 & 2 \\
0 & 186 & -62 & 62 & 84 & -45 & 3 & 0 & 186 & -62 & 62 & 84 & -45 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \text{step3} & 0 & -92 & \frac{74}{3} & -\frac{74}{3} & -\frac{10}{3} & 25 & \frac{22}{3} \\
-10 & -72 & 18 & -18 & -3 & 25 & 14 & 0 & 28 & -8 & 8 & 14 & -6 & 1 \\
0 & 28 & -8 & 8 & 14 & -6 & 1 & 0 & 22 & \frac{22}{3} & \frac{22}{3} & -5 & \frac{8}{3} \\
17 & -12 & 3 & -3 & -8 & -5 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{align*}
\]
We thus find the eigenvectors $S$ and would simply produce a different matrix letting $v$ (geometric multiplicity 2), and we choose two linearly independent eigenvectors other components are then determined again. The eigenspace is therefore 2-dimensional.

We try to find further vectors by solving $(A - 2I)v = 0$, $(A - 2I)v = 0$, $(A - 2I)v = 0$. And then, we try to solve $(A + 2I)v' = v$. We therefore find two eigenvectors: $v(7)$ can be chosen arbitrarily, but the 5th eqn requires $6v(6) - v(7) = 0$. The 4th eqn then determines $v(5)$, whereas $v(4)$ is again arbitrary. The other components are then determined again. The eigenspace is therefore 2-dimensional (geometric multiplicity 2), and we choose two linearly independent eigenvectors $v$ and $w$ by letting $v(7) = 2, v(4) = 0$, and $w(7) = 0, w(4) = 1$. Other choices would be equally legitimate and would simply produce a different matrix $S$.

We thus find the eigenvectors $v = [\frac{1}{3}, \frac{1}{3}, 0, 0, -1, \frac{1}{3}, 2]^T$ and $w = [0, 0, 1, 1, 0, 0, 0]^T$. The system $(A + 2I)v' = v$ has the solutions $v' = [0, \frac{1}{12}, 0, 0, 0, 0]^T + c_1 v + c_2 w$. The most convenient choice is of course $c_1 = c_2 = 0$, and we have $v' = [0, \frac{1}{12}, 0, 0, 0, 0]^T$.

We try to find further vectors by solving $(A + 2I)v'' = v'$ and $(A + 2I)w'' = w'$ (if possible). We wouldn’t know yet if any particular among these equations has a solution, but the JNF theorem guarantees that altogether we find three solutions, i.e., among the tentative vectors $\{v', v'', w', w'', \ldots\}$ three will actually exist (such as to bring the total to 5 generalized eigenvectors, according to the algebraic multiplicity 5).

I skip the remaining linear systems (the row transformations could be reused) and merely give the results:

The system $(A + 2I)v' = v$ has the solutions $v' = [2, \frac{3}{2}, 0, 0, -1, 5, 0]^T + c_3 v + c_4 w$, and again we choose $v' = [2, \frac{3}{2}, 0, 0, -1, 5, 0]^T$. We therefore find three solutions, i.e., among the tentative vectors $\{v', v'', w', w'', \ldots\}$ three will actually exist (such as to bring the total to 5 generalized eigenvectors, according to the algebraic multiplicity 5).
The system \((A + 2I)v'' = v'\) has the solutions \(v'' = [\frac{1}{3}, \frac{1}{3}, -\frac{1}{6}, 0, \frac{3}{4}, \frac{3}{2}]^T + c_5 v + c_6 w\).

We choose \(c_5 = c_6 = 0\).

The system \((A + 2I)w'' = w'\) has NO solutions.

We have already 5 vectors, and indeed the system \((A + 2I)v'' = v''\) (which we might yet consider trying) has no solutions. At this moment we know that the two Jordan blocks for \(\lambda = -2\) have sizes 3 and 2 respectively, corresponding to the sets of vectors \(\{v_1 := v, v_2 := v', v_3 := v''\}\) and \(\{v_4 := w, v_5 := w'\}\). (Now that we know the sizes of the Jordan blocks, we can number them sensibly.)

We still need eigenvectors \(v_6\) and \(v_7\) for the eigenvalues \(i\) and \(-i\) respectively. Then we put them all as columns in our matrix \(S\). We get \(AS = SJ\) with:

\[
S = \begin{bmatrix}
\frac{1}{3} & 0 & \frac{1}{3} & 0 & 2 & -1 & -1 \\
\frac{1}{12} & \frac{1}{3} & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{6} & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & -\frac{3}{4} & 0 & -1 & -1 + i & -1 - i \\
\frac{1}{3} & 0 & \frac{1}{3} & 0 & 5 & -\frac{9 + 8i}{5} & \frac{9 - 8i}{5} \\
2 & 0 & \frac{3}{2} & 0 & 0 & 1 - 4i & 1 + 4i
\end{bmatrix}
\]

\[
J = \begin{bmatrix}
-2 & 1 & 0 \\
0 & -2 & 1 \\
0 & 0 & -2
\end{bmatrix}
\]

\[
\lambda = \begin{bmatrix}
\lambda \\
\lambda^2 \\
\lambda^n
\end{bmatrix}
\]

\(S\) is invertible as predicted by the general theory. I won’t write down \(S^{-1}\), but note that \(\det(A + 2I) = \det(SJS^{-1})\).

\(\ker(A + 2I)\) is spanned by columns 1 and 4 of \(S\), \(\ker(A + 2I)^2\) is spanned by columns 1,2,4,5 of \(S\). The complete eigenspace for eigenvalue \(-2\) is \(\ker(A + 2I)^3 = \ker(A + 2I)^j\) for any \(j \geq 3\), spanned by columns 1–5 of \(S\). — Have a look at the (3,4) entry of \(J\), which is 0. It is this 0 that distinguishes between the case of a 3-Jordan block and a 2-Jordan block, as opposed to a single 5-Jordan block.

The span of columns 6 and 7 of \(S\) can also be spanned by real vectors (which are then no longer eigenvectors), namely the real and imaginary parts of these two vectors, \([-1, 0, 0, 0, -1, -\frac{9}{5}, 1]^T\) and \([0, 0, 0, 0, 0, 0, \frac{8}{5}, -4]^T\).

### Exponentials

The exponential of a Jordan block of size \(n\) can be calculated explicitly:

\[
\exp(t \lambda I + t) = \exp\left(t \lambda I + t \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = e^{t \lambda} \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n-1}}{(n-1)!} \\
1 & t & \cdots & \frac{t^n}{n!} \\
0 & 1 & \cdots & \frac{t^{n-1}}{(n-2)!} \\
0 & 0 & \cdots & 1 \\
0 & 0 & \cdots & 0 \end{bmatrix}
\]

Therefore the JNF theorem implies that \(\exp(tA) \to 0\) as \(t \to \infty\) if and only if all eigenvalues of \(A\) have \(\text{Re}\, \lambda < 0\).

Moreover, \(\exp(tA)\) remains bounded as \(t \to \infty\) if and only if all eigenvalues of \(A\) have \(\text{Re}\, \lambda \leq 0\) and all Jordan blocks for eigenvalues with \(\text{Re}\, \lambda = 0\) have size 1.