1. Which of the following are inner products on $\mathbb{R}^3$? For those that are not, determine which of the properties does/do not hold.
   (a) $\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 3u_2v_2$
   (b) $\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 3u_2v_2 + 3u_3v_3 - 2u_2v_3 - 2u_3v_2$
   (c) $\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 3u_2v_2 - u_3v_3$
   (d) $\langle \mathbf{u}, \mathbf{v} \rangle = (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2)$

2. If $A$ is an invertible $n \times n$ matrix, then the formula $\langle \mathbf{u}, \mathbf{v} \rangle := (A\mathbf{u}) \cdot (A\mathbf{v})$ gives an inner product, which is called the inner product generated by $A$. Calculate the inner product generated by $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ of the vectors $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

3. In this problem, let $\|\mathbf{w}\|$ denote not the euclidean norm, but the norm obtained from the inner product generated by the matrix $A$ from the previous problem.
   What condition do $x$ and $y$, the components of the vector $\mathbf{w}$ have to satisfy such that $\|\mathbf{w}\| = 3$ holds? Draw the curve $\|\mathbf{w}\| = 3$ in the plane in a good, to-scale figure. If $\|\mathbf{w}\|$ were the euclidean norm, then the curve would be a circle of radius 3. What kind of curve did you come up with in the present case? Suggested plot window: $x \in [-8, 8]$, $y \in [-5, 5]$.

4. On the space $\mathbb{R}^{3 \times 3}$, we take the inner product $\langle A, B \rangle := \text{tr} A^T B$. Based on this inner product, what is the norm of the matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$?
   Before calculating too much, remember that in a certain matrix product you are about to compute, not all of the entries need to be computed.

5. To show that $\text{tr} AB$ is not an inner product, even for square matrices, find an example of a $2 \times 2$ matrix $A$ for which $\text{tr}(A^2) < 0$.

6. In the vector space $C[0, 2\pi]$ of all continuous functions defined on the interval $[0, 2\pi]$, we use the inner product $\langle f, g \rangle := \int_0^{2\pi} f(x)g(x) \, dx$. Write out explicitly what the Cauchy-Schwarz inequality says in the case $f(x) = x$ and $g(x) = \sin x$.
   Also check this particular instance of the Cauchy-Schwarz inequality by explicit evaluation of all integrals. — I do expect Math, Phys and Engr majors to be able to do these integrals without help, but since this is not a calculus class, and for the benefit of those who have not used calculus for a while, I give some derivative-antiderivative formulas for your convenience:
   $$\frac{d}{dx}(\sin x - x \cos x) = x \sin x, \quad \frac{d}{dx} \frac{1}{2}(x - \sin x \cos x) = \sin^2 x$$

7. Continuing the previous problem, what is the “angle” between the functions $f$ and $g$, in the sense determined by the inner product given there. — Note: If you have even the slightest gut feeling that this “angle” has anything to do with any angle you might see when you graph the function $f$ and $g$, be assured emphatically that this is not the case.
8. For \( m = 0, 1, 2, 3, \ldots \), we define the functions \( c_m \in C^0[0, 2\pi] \) by \( c_m(x) := \cos mx \). For \( m = 1, 2, 3, \ldots \), we define the functions \( s_m \in C^0[0, 2\pi] \) by \( s_m(x) := \sin mx \). We share this workload among the class. Everybody select at random one of the three (equally easy/difficult) parts as homework.

(a) Show that \( \langle c_m, c_n \rangle = 0 \) if \( m \neq n \).
(b) Show that \( \langle s_m, s_n \rangle = 0 \) if \( m \neq n \).
(c) Show that \( \langle c_m, s_n \rangle = 0 \).

You may use the following trig’ formulas (which I would not expect you to have memorized):

\[
\begin{align*}
\cos u \cos v &= \frac{1}{2} (\cos(u + v) + \cos(u - v)) \\
\sin u \sin v &= \frac{1}{2} (\cos(u - v) - \cos(u + v)) \\
\sin u \cos v &= \frac{1}{2} (\sin(u + v) + \sin(u - v))
\end{align*}
\]

9. Refer to the previous problem, and have a look back at hwk 10 from Ch. 5. Show that the set \( S = \{c_0, c_1, c_2, \ldots, c_{100}, s_1, s_2, \ldots, s_{100}\} \) in \( C^0[0, 2\pi] \) is linearly independent. — What is the dimension of \( C^0[0, 2\pi] \)? (Possible answers are either a nonnegative integer, or “infinite-dimensional”.)

10. In \( \mathbb{R}^4 \), with the euclidean inner product (dot product), let \( U \) be the subspace spanned by the vectors \([1, 1, 1, 1]^T\) and \([-1, 0, 1, 2]^T\). Find a basis for the orthogonal complement \( U^\perp \) of \( U \).

11. Same question, but this time, take the inner product \( \langle u, v \rangle := u_1v_1 + 2u_2v_2 + 3u_3v_3 + u_4v_4 \) instead of the dot product.

(’I’d like to stress that the \( U^\perp \) in this problem refers to a different space than the \( U^\perp \) in the previous problem, because the meaning of the symbol \( \perp \) depends on the inner product chosen. This is why I give two so similar problems)

12. In \( \mathbb{R}^3 \) with the dot product, use the Gram–Schmidt process to obtain an orthonormal basis from the basis \( S = \{[2, 2, 1]^T, [1, 2, 2]^T, [2, -1, 2]^T\} \).

13. In the vector space \( P_3 \) of polynomials of degree 3 or less, choose the inner product \( \langle p, q \rangle := \int_{-1}^{1} p(x)q(x) \, dx \). Use the Gram–Schmidt process to obtain an orthogonal basis \( \{p_0, p_1, p_2, p_3\} \) from the standard basis \( S = \{1, x, x^2, x^3\} \). Normalize the vectors (i.e., polynomials) such that \( \|p_0\|^2 = 2, \|p_1\|^2 = \frac{2}{3}, \|p_2\|^2 = \frac{2}{5}, \|p_3\|^2 = \frac{2}{7} \).

(’Note: This is a convenient normalization that avoids square roots in the coefficients.
— The polynomials you obtain are called Legendre polynomials, and the reason why they deserve to be given a name is exactly that they form an orthogonal basis. — You are not required to memorize their name, though. — Here is one piece of information about the polynomials \( p_j \) that I provide for the sole purpose that you can check your calculation in each step against possible calculational errors: \( p_j(1) = 1 \) for all \( j \).)

14.

15.

16.

17.