Calculus Basics for M572

Continuous Functions: Let $\Omega \subseteq \mathbb{R}^n$ be an open set, then $C^k(\Omega)$ is the set of all functions defined on $\Omega$ such that the function and its first $k$ derivatives are continuous on $\Omega$. If $k = 0$ then $\Omega$ can be a closed (or partially closed) set. If $\Omega$ is partially open then without further assumptions we cannot assume that if $f \in C^k(\Omega)$ then $f$ or any derivative of $f$ up to $k$ is bounded on $\Omega$. Example: $f(x) = \sqrt{x}$ is in $C([0, 1])$, $C([0, \infty))$ and $C^\infty((0, \infty))$. $f'$ is not in $C([0, 1])$ as $f'$ does not exist at $x = 0$.

Taylor’s Theorem: Let $a \in \mathbb{R}$ and let $I$ be an open interval containing $a$. Let $f \in C^{n+1}(I)$. Then for all $x \in I$, $f(x) = P_n(x) + R_n(x)$ where

$$P_n(x) = f(a) + f'(a) \cdot (x-a) + \frac{1}{2} f''(a) \cdot (x-a)^2 + \ldots + \frac{1}{n!} f^{(n)}(a) \cdot (x-a)^n$$

$$= \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a) \cdot (x-a)^k;$$

and

$$R_n(x) = \int_a^x \frac{1}{n!} (x-t)^n f^{(n+1)}(t) \, dt = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(z),$$

where $z$ is between $a$ and $x$. Note that $R_n(a) = 0$ by definition. $P_n$ is called the Taylor Polynomial of degree $n$ for $f$ centered at $x = a$ and $R_n$ is called the Remainder.

Taylor’s Theorem is used in both the development and analysis of numerical methods. When we replace $f$ in an expression by $P_n + R_n$, the resulting expression involving $P_n$ often leads to a method, while the part involving $R_n$ leads to a term for the error in the method. In analyzing the error from the Remainder term it is important to be able to bound the error. Typically this involves finding a value $M \geq 0$ such that $|f^{(n+1)}(z)| \leq M$ for all $z \in I$. Such a $M$ can be found by careful analysis involving derivatives etc. or, since it is just a constant, can be found by breaking an expression into parts and bounding each part.

Mean Value Theorem: (This can be thought of as a special case of Taylor’s Theorem)
Let $a < b$, $f \in C([a, b])$ and $f' \in C((a, b))$. Then there is a $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Mean Value Theorem for Integrals: Let $a < b$ and $g, h \in C([a, b])$. Suppose that $h$ does not change sign in the interval $[a, b]$ then there exists a $c \in (a, b)$ such that

$$\int_a^b g(t) h(t) \, dt = g(c) \int_a^b h(t) \, dt.$$

If $h$ does change sign then often an application of Taylor’s Theorem can help things out. For example, let $a = 0$, $b = 1$ and $h(t) = t - \frac{1}{2}$. Then if $g \in C^1(a, b)$ then we can write
\[ g(t) = g\left(\frac{1}{2}\right) + g'(z)(t - \frac{1}{2}) \] for some \( z \) (depending on \( t \)) and so

\[
\int_0^1 g(t) h(t) \, dt = \int_0^1 g\left(\frac{1}{2}\right) h(t) \, dt + \int_0^1 g'(z(t))(t - \frac{1}{2})^2 \, dt
\]

\[
= 0 + g'(z(c)) \int_0^1 (t - \frac{1}{2})^2 \, dt,
\]

where \( 0 < c < 1 \) since \((t - \frac{1}{2})^2\) does not change sign in \([0, 1]\).

**Taylor’s Theorem in \( m \) variables:** Many results for 1 variable don’t extend into two or more variables. However, Taylor’s Theorem does. Let \( a \in \mathbb{R}^m \) and let \( \Omega \subseteq \mathbb{R}^m \) be an open set containing \( a \). Let \( f \in C^{n+1}(\Omega) \). Then for every \( x \in \Omega \) such that the line segment from \( a \) to \( x \) is in \( \Omega \), there is a point \( z \) on that segment such that

\[
f(x) = P_n(x) + R_n(x)
\]

\[
f(x) = \sum_{k=0}^{n} \frac{1}{k!} D^k f(a) (x - a)^k + \frac{1}{(n+1)!} D^{n+1} f(z) (x - a)^{n+1},
\]

where

\[
D^k f(a)(x - a)^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \frac{\partial^k f(a)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_m^{\alpha_m}} (x_1 - a_1)^{\alpha_1} \cdots (x_m - a_m)^{\alpha_m}.
\]

Note \( \alpha \) is a multiindex where \( \alpha \in \mathbb{Z}^m, \alpha_i \geq 0, |\alpha| = \sum_{i=1}^{m} \alpha_i \) and \( \alpha! = \alpha_1! \alpha_2! \cdots \alpha_m! \).

In the case of just 2 variables \((x, y)\), we have

\[
D^k f(a)(x - a)^k = \sum_{i=0}^{k} \frac{k!}{i!(k - i)!} \frac{\partial^k f(a)}{\partial x^i \partial y^{k-i}} (x - a_1)^i (y - a_2)^{k-i}.
\]

And we get, for \( n = 2 \),

\[
P_n(x, y) = f(a) + \frac{\partial f(a)}{\partial x} (x - a_1) + \frac{\partial f(a)}{\partial y} (y - a_2) + \frac{1}{2} \frac{\partial^2 f(a)}{\partial x^2} (x - a_1)^2
\]

\[
+ \frac{\partial^2 f(a)}{\partial x \partial y} (x - a_1)(y - a_2) + \frac{1}{2} \frac{\partial^2 f(a)}{\partial y^2} (y - a_2)^2.
\]