You can use any written resources, but please work alone.

15. Compute the LU Decomposition of

\[
A = \begin{pmatrix}
1 & 1 + \delta & 1 \\
1 & 1 & 1 + \delta \\
1 & 1 & 1
\end{pmatrix},
\]

and use the decomposition to solve \(Ax = b\) where \(b = (2, 2, 2 - \delta)^T\).

16. Let \(L_1\) and \(L_2\) be unit lower triangular \(m \times m\) matrices, i.e. ones on the diagonal and zeros above the diagonal.

(a) Show that \(L_1L_2\) is unit lower triangular.

(b) Show that \(L_1^{-1}\) exists and is unit lower triangular.

17. Let \(A \in \mathbb{C}^{m \times m}\), with \(\det A \neq 0\). Assume that \(A\) has an (Doolittle) LU-decomposition without row interchanges, i.e. \(A = LU\). Show that \(L\) and \(U\) are unique. As an example when \(\det A = 0\), give two different LU-decompositions of the matrix

\[
A = \begin{pmatrix}
0 & 1 \\
0 & 1
\end{pmatrix}.
\]

18. Do either part (a) or part (b).

(a) Consider the \(m \times m\) matrix which arises while solving a discretization of the 1D Poisson equation:

\[
A = \begin{pmatrix}
2 & -1 & 0 & \ldots & 0 \\
-1 & 2 & -1 & 0 & \ldots \\
0 & -1 & 2 & -1 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & -1 & 2 \\
0 & \ldots & 0 & 0 & -1 \\
0 & \ldots & 0 & 0 & 0 \\
\end{pmatrix}.
\]

i. Find the LU-decomposition of \(A\) for general \(m\).

ii. Compute \(\|A\|\) using whatever matrix norm you want.

iii. Show that \(\|A^{-1}\| \geq (2 - 2 \cos \frac{\pi}{m+1})^{-1}\). Hint: Compute \(Ax\) with \(x = (x_j)\), \(x_j = \sin \frac{\pi j}{m+1}\).

iv. Conclude from 18(a)(ii) and 18(a)(iii) that \(\kappa(A) \geq Cn^2\) for some constant \(C\).

(b) Let \(A \in \mathbb{C}^{m \times m}\) be a banded matrix with bandwidth \(p\), i.e. \(A_{ij} = 0\) if \(|i - j| > p\). Write versions of LU-decomposition and forward and backward substitution (pseudo-code) for solving \(Ax = b\) which takes into account this structure. Determine a formula for the number of flops it would take to solve \(Ax = b\) using this algorithm in terms of \(p\) and \(m\).