You can use any written resources, but please work alone.

5. Let $\| \cdot \|$ be a (vector) norm on $\mathbb{C}^m$. Prove that the matrix norm induced by $\| \cdot \|$ is a (vector) norm on $\mathbb{C}^{m \times m}$. In other words, show that the induced matrix norm satisfies

(a) For all $A$, $\| A \| \geq 0$.
(b) $\| A \| = 0$ if and only if $A = 0$.
(c) $\| \alpha A \| = |\alpha| \| A \|$ for any scalar $\alpha$.
(d) $\| A + B \| \leq \| A \| + \| B \|$, for all $A, B$.

6. Let $\| \cdot \|$ denote any vector norm on $\mathbb{C}^m$ and also the induced matrix norm on $\mathbb{C}^{m \times m}$. Let $A \in \mathbb{C}^{m \times m}$.

(a) Show that if $\lambda$ is any eigenvalue of $A$ then $|\lambda| \leq \| A \|$.
(b) (#3.2, pg. 24) Use (a) to show that $\rho(A) \leq \| A \|$.

Recall: $\rho(A)$ is the spectral radius of $A$, and is defined by $\rho(A) = \max \{|\lambda| : \lambda$ is an eigenvalue of $A\}$.

7. (a) Prove that for any diagonal matrix $D \in \mathbb{C}^{m \times m}$ there is a sequence of invertible diagonal matrices $\{D_n\}$ such that $\lim_{n \to \infty} \| D - D_n \|_2 = 0$.

(b) (#5.2, pg. 37) Prove, using the SVD (and part (a)), that for any $A \in \mathbb{C}^{m \times n}$ there is a sequence of matrices of full rank $\{A_n\}$ such that $\lim_{n \to \infty} \| A - A_n \|_2 = 0$.

This shows that full rank matrices are dense in $\mathbb{C}^{m \times n}$. 