1. For the general solution of the system:
\[
\frac{dx}{yz} = \frac{dy}{-xz} = \frac{dz}{xy(x^2 + y^2)},
\]
we start with the first two terms and after eliminating the common z term, get \(dx/y = -dy/x\). Assuming \(y \neq 0\) (e.g. \(y > 0\)), this is equivalent to \(dy/dx = -x/y\) which has the solution \(x^2 + y^2 = C\). Now use the first and third term and eliminate the common y term to get \(dz/z = dy/x(x^2 + y^2)\). To eliminate the further dependence on \(y\), use the first solution and get \(dz/z = dz/Cx\). Thus assuming \(z \neq 0\), this ODE is equivalent to \(dz/dx = Cx/z\). Solving this we get \(z^2 = Cx^2 + D\) or \(z^2 = (x^2 + y^2)x^2 + D\). Thus the general solution is the curve defined by the conditions:
\[
x^2 + y^2 = C \text{ and } z^2 - (x^2 + y^2)x^2 = D,
\]
with \(y > 0\). To check this, we can parameterize this curve in terms of \(t\) to get \(x = R\cos t\), \(y = R\sin t\) and \(z^2 = R^2\cos^2 t + D\) where \(C = R^2\). If we then compute \(dx/dt\), \(dy/dt\) and \(dz/dt\), we find that \((dx/dt)/(yz) = (dy/dt)/(-xz) = (dz/dt)/(xy(x^2 + y^2)) = -1/z\).

2. The general solution of \(u_x = 0\) is \(u(x, y, z) = F(y, z)\) from direct ‘partial’ integration. If we look for integral solutions from \(dx/1 = dy/0 = dz/0\) we get \(u_1 = y\) and \(u_2 = z\) from combining the first and second, and the first and third terms, respectively.

The general solution of \(yu_x - xu_y = 0\) comes from the integral solution of \(dx/y = dy/(-x) = dz/0\). Assuming \(y \neq 0\), this is equivalent to the two ODES: \(dy/dx = -x/y\) and \(dz/dx = 0\). From the first we get \(u_1 = x^2 + y^2\) and the second \(u_2 = z\). Thus the general solution is \(u(x, y, z) = F(x^2 + y^2, z)\).

Note that on the curve defined by \(x^2 + y^2 = R^2\) and \(z = C\) the solution takes on only one value: \(u(x, y, z) = F(R^2, C)\).

3. (a) For \(zz_x + yz_y = x\), we have \(dx/z = dy/y = dz/x\). Setting the 1st and 3rd terms equal, assuming \(x \neq 0\) and solving, we get we get we get \(z^2/2 = x^2/2 + c\) and the first integral \(u_1(x, y, z) = x^2 - z^2\). Using the 1st and 2nd terms and substituting \(z = \sqrt{x^2-c}\), we get \(dx/\sqrt{x^2-c} = dy/y\). Integrating, we get \(x+z = Cy\) or \((x+z)/y = C\). So \(u_2(x, y, z) = (x+z)/y\).

We can also get this integral by combining \(dx/dt = z\) with \(dz/dx = x\) to get \(d(x+z)/dt = (x+z)\) and setting \(w = x+z\), we have the pair \(dw/w = dy/y\) which give \(w/y = D\) or \((x+z)/y = D\).

Some examples of general integrals: \(u(x, y, z) = F(u_1, u_2) = 0\):
\[
u_1 - a^2 = 0: z = \sqrt{x^2-a^2}, \text{ domain either } x \geq a \text{ or } x \leq -a.
\]
\[
u_2 - a = 0: z = ay - x, \text{ domain all } x, y
\]
\[
u_1u_2 - a = 0: z = x - a/y, \text{ domain either } y > 0 \text{ or } y < 0.
\]

(b) For \(x^2z_x + y^2z_y = (x+y)z\), we have \(dx/x^2 = dy/y^2 = dz/(x+y)z\). Setting the 1st and 2nd terms equal and integrating, we get \(-1/x = -1/y + c\), so \(u_1(x, y, z) = 1/x - 1/y\). Setting the 1st and 3rd together and using \(y = x/(1-cx)\) and integrating, we get \(z(cx+1)/cx^2 = d\), or subbing back in for \(c\), we get \(u_2 = z/(x-y)\). We could also combine the 1st and 2nd terms to get \(d(x-y)/(x^2-y^2) = dz/(x+y)z\) and then by simplifying, get \(d(x-y)/(x-y) = dz/z\) which has the solution \((x-y)/z = D\).

Some general integrals:
\[
u_1 + u_2 = 0: z = (x-y)/(xy), \text{ domain one of four quadrants}
\]
\[
u_2 - a = 0: z = a(x-y), \text{ domain all } x, y
\]
\[
u_1u_2 + a = 0: z = axy, \text{ domain all } x, y
\]
(c) For $z_y = 3y^2$, we have $dx/0 = dy/1 = dz/3y^2$. So $u_1 = x$, and setting the 2nd and 3rd terms equal, we get $u_2 = y^3 - z$. General integrals:

\[ f(u_1) + u_2 = 0: z = f(x) + y^3, \text{ domain all } x \text{ in the domain of } f \]

Note that since $u = F(u_1, u_2)$ must satisfy $u_z \neq 0$, we must have $\partial F/\partial u_2 \neq 0$ so we can solve $F(u_1, u_2) = 0$ for $u_2$ and thus every general integral must have the form $z = f(x) + y^3$ for some $f$.

4. (a) $z_x + z_y = z, z = \cos(t)$ on $x = t, y = 0$. The general integral is $u(x, y, z) = G(x - y, z e^{-y})$. Combine $x - y = t$ and $z e^{-y} = \cos(t)$ to make zero, we see that $\cos(x - y) - z e^{-y}$ works, which leads to the solution $z = e^y \cos(x - y)$. The domain is all $x, y$.

(b) $xz_x - yz_y = 0, z = x^2$ on $y = x, x > 0$. The general integral is $u(x, y, z) = G(xy, z)$ or as in the discussion of the previous problem where only one component has $z$ in it, we have that $z = f(xy)$. Applying the initial value, we see that $f(t) = t$ works and so $z = xy$ is our solution, with domain $x, y > 0$ since we needed to make that restriction in finding $u_1$. 