1. Newton’s Law of Cooling (and Warming): The rate of change of the temperature of a body is proportional to the difference between the body’s temperature and the temperature of the surrounding medium.

(a) Write an ODE model for the body’s temperature $b(t)$ in terms of the medium’s temperature $m(t)$ where the constant of proportionality is $k$ (called the coefficient of heat transfer). Note that if $b(t) > m(t)$ then $b'(t) < 0$.

(b) A dead body is found in a $72^\circ$ room. At 6PM the temperature of the body was $85^\circ$. An hour later the temperature was $82^\circ$. At what time did the person die?

(c) The outdoor temperature (in degrees) varies like $72 + 12 \cos(\pi t)$ where $t$ is in hours. Suppose a room is heated (and cooled) only by its contact with the outside air. Find the solution for the temperature inside the room at time $t$. How does the long-term behavior of the temperature depend on the initial conditions? How does it depend on the coefficient of heat transfer $k$?

2. Consider the homogeneous (sometimes called undriven) ODE: $y' + 2ty = 0$.

(a) Find the solution of this ODE and describe what the possible results are as $t \to \infty$.

(b) For the non-homogeneous (driven) version: $y' + 2ty = q(t)$ describe the results as $t \to \infty$ for:

(i) $q(t) = 1$,   
(ii) $q(t) = 1 + t^2$

3. Fish Growth Model with Harvesting. A simple model for fish growth with harvesting is $y' = \frac{1}{12}y(12 - y) - H(t)$ where $y(t)$ is the total tonnage of fish after $t$ years and $H(t)$ is the harvesting function.

(a) (Constant Harvesting) Solve for $y(t)$ when $H(t) = H_0$ (constant) and $y(0) = y_0$.

(b) (Two-Level Harvesting) Solve for $y(t)$ when $H(t) = H_0$ for $t < 5$ and $H(t) = H_1$ for $t \geq 5$. Hint: Use the solution from part (a) for $0 < t < 5$ then use the value of the solution at $t = 5$ as the initial condition for the part when $t > 5$.

4. Consider the predator-prey system

$$x' = x(a - by), \quad y' = -y(c - dx),$$

where $x(t)$ is the population of the prey at time $t$, $y(t)$ is the population of the predator at time $t$, and $a, b, c, d$ are positive constants.

(a) Make the substitution $x = AX + c/d, y = BY + a/b$ and $t = CT$.

(b) Choose values for $A, B$ and $C$ to make the system as simple as possible, i.e. try to reduce the model to having just one parameter instead of 4. What are the units for $A, B$ and $C$?

(c) Assume $X$ and $Y$ are small (so that $X^2, XY, Y^2$ are ignorable) and simplify the system by dropping ignorable terms. (This should result in a simple system).

(d) Now, divide the expression for $Y'$ by the expression for $X'$. This results in an expression for $\frac{dY}{dX}$. Find a solution for this differential equation which satisfies $X(0) = x_0$ and $Y(0) = y_0$.

5. Consider the model $\frac{dx}{dt} = f(x)$ where $f(x) = ax(x - b)(c - x)$ and $a, b$, and $c$ are positive constants with $b < c$.

(a) Interpret this model as a model of population growth. Specifically, assume that $x(t)$ is the level of a population, then assign some meaning to the quantities $a, b$ and $c$, and give an explanation as to why the model would have the form it does.

(b) Describe the behavior of $x(t)$ for $t > 0$ if (i) $0 < x(0) < b$, (ii) $b < x(0) < c$, and (iii) $c < x(0)$. 

1