In the exercises that follow, $A$, $B$, $C$ and $D$ are arbitrary sets, unless described otherwise.

167. If $A = (-\infty, 0]$, $B = [0, \infty)$, and $C = (0, 1)$, find and draw each of the following sets: $A \times (B \cap C)$, $(A \cap B) \times C$.

168. If $A = (-\infty, 0]$, $B = [0, \infty)$, and $C = (0, 1)$, find and draw each of the following sets: $(A \setminus B) \times (A \cap B)$, $(A \times B) \cap (B \times C)$.

169. Prove if $A \subseteq B$ and $C \subseteq D$ then $A \times C \subseteq B \times D$.

170. Prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

171. Prove that $A \times \emptyset = \emptyset$.

172. Prove that $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$.

173. Prove that $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

174. Prove or disprove that $(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$.

175. Prove that $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$.

176. Let $B \subseteq A$ and prove that $(A \times A) \setminus (B \times B) = ((A \setminus B) \times A) \times (A \times (A \setminus B))$.

177. Let $C \neq \emptyset$ and prove that if $A \times C = B \times C$ then $A = B$.

178. Let $A \neq B$ and prove that if $A \times C = B \times C$ then $C = \emptyset$.

179. Find the domain and range of the following relation: $x \, R \, y$ means $x, y \in \mathbb{R}$ and $y = 3 \cdot (x - 1)^2 + 2$.

180. Find the domain and range of the following relation: $x \, R \, y$ means $x, y \in \mathbb{R}$ and $y = 2 \cdot x^2 - 8 \cdot x + 9$.

181. Find the domain and range of the following relation: $x \, R \, y$ means $x, y \in \mathbb{R}$ and $x^2 - y^2 = 1$.

182. Find the domain and range of the following relation: $x \, R \, y$ means $x, y \in \mathbb{R}$ and $xy = x^2 - 1$.

183. Find the domain and range of the following relation: $x \, R \, y$ means $x, y \in \mathbb{R}$ and $\sqrt{1 - x} = (x + 1)y$.

184. For each of the following relations, prove or find a counterexample for each of the properties: reflexive, symmetric and transitive.

   (a) Define $x \, R \, y$ on $\mathbb{Z} \times \mathbb{Z}$ to mean $x - y$ is a nonnegative integer.

   (b) Define $x \, R \, y$ on $\mathbb{N} \times \mathbb{N}$ to mean $xy$ is an integer.

   (c) Define $x \, R \, y$ on $\mathbb{Z} \times \mathbb{Z}$ to mean $x - y$ is an odd integer.

   (d) Define $x \, R \, y$ on $\mathbb{R} \times \mathbb{R}$ to mean $x - y$ is irrational.

   (e) Define $x \, R \, y$ on $\mathbb{R} \times \mathbb{R}$ to mean $x \geq y$.

185. For each of the following relations prove that it is an equivalence relation.

   (a) Given $x, y \in \mathbb{R}$, $x \, R \, y$ means $y = \sqrt{x^2}$.

   (b) Given ordered pairs $(a, b), (p, q) \in \mathbb{R} \times \mathbb{R}$, $(a, b) \, R \, (p, q)$ means $pb = aq$. 

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Math 300 – Problems #6 - COMPLETE
(c) Given \( x, y \in \mathbb{R} \), \( x R y \) means \( x - y \) is an integer.
(d) Given \( x, y \in \mathbb{R} \setminus \{0\} \), \( x R y \) means \( x/y = \pm 1 \).
(e) Let \( n \) be a positive natural number. Define a relation \( R \) on the set of integers \( \mathbb{Z} \) by: \( x R y \) if and only if \( x - y \) is divisible by \( n \).
(f) Define a relation \( R \) on the set of integers \( \mathbb{Z} \) as follows: \( x R y \) if and only if \( x + y \) is even.
(g) Define a relation \( R \) on the set of reals as follows: \( x R y \) if and only if \( x - y \) is rational.
(h) A set \( X \) is the union of 6 subsets \( A_1, \ldots, A_6 \) which are mutually disjoint (that means \( A_i \cap A_j = \emptyset \) if \( i \neq j \)). Define the relation \( R \) on \( X \) as follows: \( x R y \) if and only if there is \( i \) so that \( A_i \) contains both \( x \) and \( y \).

186. Let \( R \) be a relation on \( X \times Y \). Given \( x \in X \) we define the subset \( A_x \) of \( Y \) as follows: \( A_x = \{ y \in Y \mid x R y \} \). Prove that if \( R \) is an equivalence relation (and \( X = Y \)) then for each pair \( x, y \in X \) then either \( A_x = A_y \) or \( A_x \cap A_y = \emptyset \) (identical or disjoint).

187. A set \( X \) contains 9 subsets \( A_1, \ldots, A_9 \) not necessarily disjoint. We define the relation \( R \) on \( X \) as follows: \( x R y \) if and only if there is \( i \) so that \( A_i \) contains both \( x \) and \( y \). Prove that \( R \) is a reflexive relation if and only if \( X \) is the union of all of \( A_i, i \leq 9 \).

188. A set \( X \) contains 9 subsets \( A_1, \ldots, A_9 \). None of them is contained in the union of remaining ones. Define the relation \( R \) on \( X \) as follows: \( x R y \) if and only if there is \( i \) so that \( A_i \) contains both \( x \) and \( y \). Prove that \( R \) is a transitive relation if all of \( A_i, i \leq 9 \) are mutually disjoint.

189. Suppose \( R \) is a symmetric relation on \( X \times X \) which satisfies

\[
\text{Domain}(R) \cup \text{Range}(R) = X.
\]

Prove that \( \text{Domain}(R) = X = \text{Range}(R) \).

190. Suppose \( R \) is a symmetric and transitive relation on \( X \times X \) which satisfies

\[
\text{Domain}(R) \cup \text{Range}(R) = X.
\]

Prove that \( R \) is an equivalence relation on \( X \).

191. Suppose \( R \) and \( S \) are equivalence relations on \( X \). Prove that \( R \cap S \) is an equivalence relation on \( X \). What about \( R \cup S \)?

192. Suppose \( R \) is a reflexive relation on \( X \times X \) with domain \( X \) which satisfies the following condition: if \( x R y \) and \( x R z \), then \( y R z \). Prove that \( R \) is an equivalence relation on \( X \). Does every equivalence relation satisfy this condition?