Math 300 – Cardinality and Infinity

**Definition:** For a set \( A \) the *cardinality* of \( A \) is the number of distinct elements in \( A \). We’ll write \( n(A) \) for the cardinality of \( A \).

**Definition:** For \( A = \{1,2,\ldots,n\} \), \( n(A) = n \). For \( \mathbb{N} = \{1,2,\ldots\} \), \( n(\mathbb{N}) = \infty \).

**Definition:** Two sets \( A \) and \( B \) have the *same size* or *cardinality* if there exists a function \( f : A \to B \) which is 1−1 and onto.

**Lemma:** Two sets \( A \) and \( B \) have the same size if there is a function \( f : A \to B \) which is 1−1 and a function \( g : A \to B \) which is onto.

**Exercises:** Prove or disprove in each case that \( n(A) = n(B) \):

1. \( A = \mathbb{N}, B = \{2,3,\ldots\} \).
2. \( A = \mathbb{N}, B = \{0,1,2,3,\ldots\} \).
3. \( A = \mathbb{N}, B = \{2,4,6,8,\ldots\} \).
4. \( A = \mathbb{N}, B = \{\ldots,-2,-1,0,1,2,\ldots\} \).
5. \( A = \mathbb{N}, B = \{\frac{a}{b} \mid a, b \in \mathbb{N}, b \neq 0\} \).
6. \( A = [0,1], B = [0,1] \cup \{2\} \).
7. \( A = [0,1], B = [0,2] \).
8. \( A = [0,1], B = [0,\infty) \).
9. \( A = [0,1], B = \mathbb{R} \).
10. \( A = [0,1], B = \mathbb{N} \).