Math 142 – Quiz 6 – Solutions

1. (a) By direct calculation,

\[
\lim_{n \to \infty} \frac{1 + 3n}{2 - 5n} = \lim_{n \to \infty} \frac{\frac{1}{n} + 3}{\frac{2}{n} - 5} = \frac{0 + 3}{0 - 5} = -\frac{3}{5}.
\]

So it is convergent with limit \(-\frac{3}{5}\).

(b) By using \(f(x)\), where \(a_n = f(n)\),

\[
\lim_{x \to \infty} \frac{x^2}{e^x} = \lim_{x \to \infty} \frac{2x}{e^x} = \lim_{x \to \infty} \frac{2}{e^x} = 0.
\]

(Obtained using L'Hôpital's Rule twice). Thus the sequence is convergent with limit 0.

(c) By comparison,

\[
\frac{n - 1}{n + 1} \leq \frac{n + \cos(n)}{n + 1} \leq 1 \quad \text{and} \quad \lim_{n \to \infty} \frac{n - 1}{n + 1} = 1
\]

so by the Squeeze Theorem, the sequence \(\left\{ \frac{n + \cos(n)}{n + 1} \right\} \) converges to 1.

2. (a) By the Ratio Test (or as a geometric series),

\[
L = \lim_{n \to \infty} \left| \frac{16 \cdot 3^{2n+2}}{7^{n+1}} / \left( \frac{16 \cdot 3^{2n}}{7^n} \right) \right| = \lim_{n \to \infty} \left| \frac{3 \cdot 3^{2n}}{7^n} \right| \left| \frac{7^n}{7^n} \right| = \frac{9}{7}.
\]

The ratio is bigger than 1, so the series is divergent. Can also show using the Divergence Test since \(\lim_{n \to \infty} a_n = \infty\).

(b) By the integral test

\[
\int_2^\infty \frac{1}{x \ln x} \, dx = \lim_{t \to \infty} \int_2^t \frac{1}{x \ln x} \, dx = \lim_{t \to \infty} \ln(\ln t) - \ln(\ln 2) = \infty.
\]

So the integral diverges and thus by the Integral Test, the series also diverges.

(c) By comparison,

\[
\lim_{n \to \infty} \frac{(n^3 - 3n + 1)/(n^5 + 2n^3)}{1/n^2} = \lim_{n \to \infty} \frac{n^5 - 3n^3 + n^2}{n^5 + 2n^3} = \lim_{n \to \infty} \frac{1 - 3n^{-2} + n^{-3}}{1 + 2n^{-2}} = 1.
\]

Since \(\sum 1/n^2\) converges (\(p\)-series, \(p = 2 > 1\)), by the Limit Comparison Test, the series converges.

(d) By the ratio

\[
L = \lim_{n \to \infty} \left| \frac{(-10)^{n+1}/(n + 1)!}{(-10)^n/n!} \right| = \lim_{n \to \infty} \frac{10}{n + 1} = 0.
\]

Since the ratio is less than 1, by the Ratio Test, the series converges.

(e) With the limit

\[
\lim_{n \to \infty} \frac{n}{\sqrt{n + 1}} = \lim_{n \to \infty} \frac{\sqrt{n}}{\sqrt{1 + 1/n}} = \infty.
\]

The limit is not 0, so, by the Divergence Test, this series diverges. Can also compare to \(\sum \sqrt{n}\) via Limit Comparison Test.
(f) We have
\[ 0 \leq \frac{\sin^2 n}{n^{3/2}} \leq \frac{1}{n^{3/2}} \]

The series \( \sum \frac{1}{n^{3/2}} \) converges (\( p \)-series, \( p = 3/2 > 1 \)) so by the (Direct) Comparison Test, the series converges.

(g) By the limit
\[ \lim_{n \to \infty} \frac{1/n - 1/n^2}{1/n} = \lim_{n \to \infty} 1 - \frac{1}{n} = 1. \]

Since \( \sum 1/n \) diverges (\( p \)-series, \( p = 1 \leq 1 \)), by the Limit Comparison Test the series diverges. Can also use the integral test with \( \int \frac{1}{x} - \frac{1}{x^2} \, dx = \ln x + \frac{1}{x} \). Can also use direct comparison with \( \frac{1}{n} - \frac{1}{n^2} > \frac{1}{n-1} \).

3. By the ratio
\[ L = \lim_{k \to \infty} \left| \frac{x^{2k+2}/((k+1)2^{k+1})}{x^{2k}/(2k^2)} \right| = \lim_{k \to \infty} \left| \frac{k}{k+1} \right| |1/2| = |x|^2/2. \]

By the Ratio Test we have convergence when \( L < 1 \) or \( |x|^2/2 < 1 \) or \( |x| < \sqrt{2} \). We have divergence when \( |x| > \sqrt{2} \). When \( |x| = \sqrt{2} \), \( x^2 = 2 \) and the terms in the series reduce to \( \frac{1}{k} \) and the resulting series is divergent. So the original series is convergent only for \( |x| < \sqrt{2} \) and is divergent otherwise.

4. (a) The sequence must converge to 0, otherwise by the Divergence test, the series would be divergent.

(b) Since for small \( x \), \( \sin(x) \leq x \), \( 0 \leq \sin(a_n) \leq a_n \) (or \( \lim_{n \to \infty} \frac{\sin(a_n)}{a_n} = \lim_{x \to 0} \frac{\sin x}{x} = 1 \)). Thus by the Comparison (or Limit Comparison) Test, \( \sum \sin(a_n) \) is convergent (to something less than 1).