1. Let $A$ be in $\mathbb{R}^{5 \times 4}$ and $\text{nullity}(A - 9I) = 2$.
   a. Must $A$ have an eigenvalue of $c = 9$? If so, how many eigenvectors will $A$ have corresponding to $c = 9$? Yes! If $[A - 9I : \mathbf{0}]$ has two free variables, then $(A - 9I)\mathbf{x} = \mathbf{0}$ has more than just the solution $\mathbf{x} = \mathbf{0}$. And so, $|A - 9I| = 0$.
      But, this means $c = 9$ is a solution to $|A - cI| = 0$... which makes 9 an eigenvalue. Two null basis vectors in solution to $[A - 9I : \mathbf{0}]$ means two eigenvectors for $c = 9$.
   b. If $d \neq 9$, then the value of $\text{nullity}(A - dI)$ cannot be 3 or greater than 3. Explain.
      If $\text{nullity}(A - dI) = 3$, then the eigenvalue $d$ has 3+ eigenvectors.
      But, this cannot be. We have 2 eigenvectors for $c = 9$ already and $A$ is in $\mathbb{R}^{5 \times 4}$... so it can have at most 4 eigenvectors (since the eigenvectors are lin ind in $\mathbb{R}^{4}$). With 2 eigenvectors for $c = 9$, $A$ can have at most two more eigenvectors.
   c. If $A$ has an eigenvalue at $c = 1$ and $c = -15$, then must $A$ be similar to a diagonal matrix? If so, what will the determinant of that diagonal matrix be?
      Yes! We know $c = 9$ has two eigenvectors. And each eigenvalue has at least 1 eigenvector (since $|A - cI| = 0$ means $[A - cI : \mathbf{0}]$ must have solutions other than $\mathbf{x} = \mathbf{0}$). So, two eigenvalues in addition to $c = 9$ means at least 2 more additional eigenvectors. At least 4 eigenvectors for a $4 \times 4$ matrix means $A$ must have exactly 4 (since that is the maximum lin ind set in $\mathbb{R}^{4}$). Thus 4 eigenvalues $A$ is similar to a diagonal matrix.

2. Let $A$ be in $\mathbb{R}^{5 \times 5}$. Then, $|A - cI| \neq c^4 + 3c^2 - 7c + 1$. Explain.
   Since $A$ is in $\mathbb{R}^{5 \times 5}$, it will have an eigenpolynomial of degree 5. That is to say, $|A - cI|$ must be a degree 5 polynomial. And, $c^4 + 3c^2 - 7c + 1$ is of degree 4.

3. The vectors $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 9 \\ -1 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ cannot be eigenvectors of any matrix in $\mathbb{R}^{3 \times 3}$. Explain.
   $\begin{vmatrix} -3 & 9 & 3 \\ 0 & -1 & -1 \\ 1 & 2 & 4 \end{vmatrix} = -3 \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} + 1 \begin{vmatrix} 9 & 3 \\ 1 & -1 \end{vmatrix}$
   $= -3(-4 + 2) + 1(-9 + 3)$
   $= -3(-2) + (-6) = 0$.
   Hence, these 3 vectors are not lin ind and cannot be eigenvectors as a result.
4. Let \( A = \begin{bmatrix} 5 & -3 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \). 

\[ \text{matrix is triangular} \]

a. Find the eigenvalues of \( A \).

\[
0 = |A - cI| = \begin{vmatrix} 5-c & -3 & -1 \\ 0 & 1-c & 0 \\ 0 & 0 & 2-c \end{vmatrix} = (5-c)(1-c)(2-c)
\]

And so eigenvalues are \( c = 5 \), \( c = 1 \) and \( c = 2 \).

b. Only knowing the eigenvalues from part a., can you say for certain if the eigenspace of \( A \) spans all of \( \mathbb{R}^3 \) or not? Explain.

Yes! Each eigenvalue has at least one eigenvector. So, we have at least 3 eigenvectors. And, all eigenvectors are linearly independent, so we must have exactly 3 eigenvectors and span all 3 dimensions of \( \mathbb{R}^3 \).

c. Find the eigenvectors of \( A \).

\[
\begin{align*}
\lambda &= 5 \\
[A - \lambda I | 0] &= \begin{bmatrix} 0 & -3 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} & \rightarrow & \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \rightarrow & \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\end{align*}
\]

Solution is \( \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} \Rightarrow \text{eigenvector is } \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.

\[
\begin{align*}
\lambda &= 2 \\
[A - \lambda I | 0] &= \begin{bmatrix} 3 & -3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \rightarrow & \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \rightarrow & \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\end{align*}
\]

Solution is \( \begin{bmatrix} x_1 \\ \frac{1}{2}x_2 \\ 0 \end{bmatrix} \Rightarrow \text{eigenvector is } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.

\[
\begin{align*}
\lambda &= 1 \\
[A - \lambda I | 0] &= \begin{bmatrix} 4 & -3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \rightarrow & \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \rightarrow & \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\end{align*}
\]

Solution is \( \begin{bmatrix} x_1 \\ \frac{1}{2}x_2 \\ x_3 \end{bmatrix} \Rightarrow \text{eigenvector is } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \) or \( \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \).

d. Find a matrix \( P \) such that \( P^{-1}A = DP^{-1} \) where \( D \) is diagonal.

\[
\begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}.
\]

\( P = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \end{bmatrix} \)

\( \text{eigenvectors} \)

e. For the \( P \) you have chosen in part d., find the diagonal matrix \( D \) that corresponds to this \( P \).

\[
\begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]