1. Let \( V \) be the space of real-valued single variable functions. Let \( W \) be the collection of all functions that have a zero at \( x = 12 \). In other words, \( W \) is all functions such that \( f(12) = 0 \). Show that \( W \) is a subspace of \( V \).

   \[
   \text{closed under } +: \text{ let } f, g \text{ be functions in } W, \text{ we need to show } f + g \text{ is also in } W. \quad \text{But } f + g = f(x) + g(x).
   \]
   \[
   \text{And so, } f(12) + g(12) = 0 + 0 = 0. \quad \text{Hence, } f + g \text{ is also in } W.
   \]

   \[
   \text{closed under } \cdot: \text{ let } c \text{ be any scalar.}
   \]
   \[
   \text{We need to show } cf \text{ is also in } W. \quad \text{But } cf = c \cdot f(x).
   \]
   \[
   \text{And so, } c \cdot f(12) = c \cdot 0 = 0. \quad \text{Hence, } cf \text{ is also in } W.
   \]

2. Let \( A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 2 \\ -1 & 2 & 1 \end{bmatrix} \).

   a. Does the column space of \( A \) (ie. \( \text{gen}\{c_1, c_2, c_3\} \)) span \( \mathbb{R}^3 \)? Explain.

   In other words, can \( c_1 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \) equal any vector in \( \mathbb{R}^3 \)?

   But, this is asking if \( A \overrightarrow{c} \) can be any vector in \( \mathbb{R}^3 \). Since \( A \) is square, this will ultimately follow if \( A \) is invertible (as \( A \overrightarrow{c} = \overrightarrow{0} \) has a solution for every \( \overrightarrow{0} \)). And, \( |A| = -1 \begin{vmatrix} 2 & -2 \\ 1 & -1 \end{vmatrix} = -1(3+6) - 2(4+3) \neq 0 \).

   Hence, \( A \) is invertible and \( A \overrightarrow{c} \) can be any vector in \( \mathbb{R}^3 \). That is to say, the column space of \( A \) spans \( \mathbb{R}^3 \).

   b. Are the column vectors of \( A \) linearly independent? Explain.

   In other words, does \( c_1 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \) have only the solution \( c_1 = c_2 = c_3 = 0 \)? But, this is asking if \( A \overrightarrow{c} = \overrightarrow{0} \) has only the solution \( \overrightarrow{c} = \overrightarrow{0} \). Since \( A \) is invertible (det \( \neq 0 \) from a), we know that \( \overrightarrow{c} = \overrightarrow{0} \) is the only solution to \( A \overrightarrow{c} = \overrightarrow{0} \). Hence, the column vectors of \( A \) are indeed linearly independent.

   c. Would these two facts coincide for any invertible matrix \( A \) in \( \mathbb{R}^{n \times n} \)? Explain.

   Yes. If \( A \) is an invertible, then \( A \overrightarrow{x} \) can be any vector in \( \mathbb{R}^n \) & \( A \overrightarrow{0} = \overrightarrow{0} \) has only the solution \( \overrightarrow{x} = \overrightarrow{0} \). In other words, if \( A \) is invertible, then the column vectors of \( A \) span \( \mathbb{R}^2 \) and are linearly independent.
3. Let \( B \) be in \( \mathbb{R}^{mxn} \). If \( n > m \), then the null space of \( B \) contains infinitely many vectors of \( \mathbb{R}^n \). Explain.

The null space is all solutions to \( Bx = 0 \). The null space is never empty as \( x = 0 \) always solves the system. Hence, the question remains if there is one and only one solution or infinitely many solutions. We know \( B \) is in \( \mathbb{R}^{mxn} \) and \( n > m \), hence, more variables than equations: \( B = \begin{bmatrix} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix} \). There can be at most one leading 1 per row; hence, there are at most \( m \) isolated variables. But, this means there is at least one free variable (since total \( n \) of variables is \( n \)).

4. The space \( \text{gen} \left[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right] \) spans diagonal matrices in \( \mathbb{R}^{3x3} \). Explain.

The space generated by these vectors is all linear combinations of these vectors. In other words, all elements of the form \( c_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \). Simplifying, this is all elements of the form \( \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix} \). These are the diagonal matrices of \( \mathbb{R}^{3x3} \).

5. Show that \( \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \), and \( \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \) are linearly independent in \( \mathbb{R}^{2x2} \).

To show, \( c_1 \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \). Simpifying, we get \( \begin{bmatrix} c_1 + 2c_2 + c_3 \\ 3c_1 + 3c_2 + 3c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \). This means \( c_1 + 2c_2 + c_3 = 0 \) and \( 3c_1 + 3c_2 + 3c_3 = 0 \). This means \( 3c_1 + 3c_2 + 3c_3 = 0 \).

This system will have only the solution \( x = 0 \) when the coefficient matrix is invertible.

6. Let \( V \) be a vector space, and let \( W \) be a subspace of \( V \). If \( W \) contains a single non-zero vector, it must contain infinitely many vectors. Explain.

Let \( \vec{x} \) be in \( W \) such that \( \vec{x} \neq \vec{0} \). Since \( W \) is a subspace, it is closed under scalar multiplication. Hence, \( c\vec{x} \) is in \( W \) for every \( c \).

But, there are infinitely many such \( c \).

**Very Rigorous proof:** If \( c_1 \neq c_2 \) and \( \vec{x} \neq \vec{0} \), then \( c_1\vec{x} - c_2\vec{x} = (c_2 - c_1)\vec{x} \neq \vec{0} \). This can't be zero since \( c_1 \neq c_2 \) means \( c_2 - c_1 \neq 0 \) and \( \vec{x} \neq \vec{0} \). Hence, their product is non-zero. And \( c_2\vec{x} - c_2\vec{x} = 0 \) means \( c_2\vec{x} = c_2\vec{x} \). In other words, different scalar multiples indeed yield different vectors.