Wed Jan 27  Lecture Commentary

Why the integral of \( v(t) \) from \( t=a \) to \( t=b \) is \( s(b)-s(a) \)

I just want to fill in a single detail from Dr. Phan’s explanation. He noted that as the number of rectangles in your partition \( N \rightarrow \infty \) that \( \Delta t \cdot s'(t_j) \approx \Delta t \cdot s'(t) \) for any \( t \) on the interval \([t_{j-1}, t_j]\)…which makes sense since as the subinterval becomes very small, the function values can only differ so much. He then said, this expression \( \Delta t \cdot s'(t) \approx \text{total change in } s(t) \) on \([t_{j-1}, t_j]\).

But, what is this total change? It is just the final position minus the initial position. So, \( s(t_j) - s(t_{j-1}) \). And, what do you get when you add up these total changes over all of your intervals?

\[
\sum_{i=1}^{N} (s(t_i) - s(t_{i-1})) = (s(t_1) - s(t_0)) + (s(t_2) - s(t_1)) + \cdots + (s(t_{N-1}) - s(t_{N-2})) + (s(t_N) - s(t_{N-1}))
\]

Note the cancellation. The terms \( s(t_1), s(t_2), \ldots, s(t_{N-1}) \) all cancel, leaving just \( -s(t_0) + s(t_N) \). In other words, \( s(t_N) - s(t_0) \ldots \text{or } s(b) - s(a) \). Hence, the value of a definite integral depends only on the value of the antiderivative at the endpoints. In our particular case, this general principle shows itself as

\[
\int_a^b v(t) \, dt = s(b) - s(a).
\]

Distance Traveled vs Displacement

Think of distance traveled as what an odometer would measure. It doesn’t matter which direction you’re going in the car, any movement in the car is recorded as positive distance traveled. Hence, why velocity (which is normally a signed measure) needs to be considered as its absolute value function when determining distance traveled. Hence, \( \text{Dist Traveled} = \int_a^b |v(t)| \, dt \).

Think of displacement as what one could measure if they only had two photo snapshots of your position, one at the initial instant and one at the final instant. If you had such a recording, say at 5am one morning and 4am the next, it is likely that your displacement would be nearly zero as you’d probably be in bed at the same location at both the initial and final instants…even though the total distance traveled by you in that 23hr period was probably significantly greater than zero. Hence, displacement does not consider all motion as positive distance. Displacement only depends on the values at the endpoints. And so, \( \text{Displacement} = \int_a^b v(t) \, dt \).

How to think of the integral of an absolute value function

Going back to basics. What does an absolute value function do, it makes whatever value the function takes positive. In particular, anywhere where the original function is negative now becomes positive. If you compute the area under the velocity function on \([a, b]\), you could do it as \( \lim_{\|P\| \to 0} \sum_{i=1}^{N} \Delta x_i \cdot v(c_i) \). Here, when the function dips below the \( t \)-axis, the \( v(c_i) \) values are negative, and hence the area computation for regions of the graph where \( v(t) \) dips below the \( t \)-axis are negative.

Inserting absolute values yields \( \lim_{\|P\| \to 0} \sum_{i=1}^{N} \Delta x_i \cdot |v(c_i)| \). This simply takes any region of area below the \( t \)-axis that was previously computed as negative and makes it positive. End of story.
So, how does one deal with this in practice?

Consider where your function takes a value of zero. Break the integral into pieces at these points. If your integral computation for any region comes out positive, you are dealing with a region above the axis. The absolute value function will see this area also as positive. But, if your integral computation yields any region to have a negative area, you need to account for that area as positive area. Add up the (forcefully positive) areas of all the regions, and you'll have it. Problem solved.

Let's see this logic demonstrated in the example in class:

The problem: \( \int_{0}^{3\pi/2} \sin(t) \, dt \).

1. Find where the function takes a value of zero. Where on \([0, \frac{3\pi}{2}]\) is \( \sin(t) = 0 \)? \( t = 0, \pi \).
2. Break the integral into pieces at these points:
   - Let \( \int_{0}^{\pi} \sin(t) \, dt \) be region 1. Let \( \int_{\pi}^{3\pi/2} \sin(t) \, dt \) be region 2.
   - (Note, in particular, we are breaking this region at \( t = \pi \), where the function is crossing the \( t \)-axis.)
3. Compute the individual integrals separately:
   - \( \int_{0}^{\pi} \sin(t) \, dt = -\cos(t) \bigg|_{0}^{\pi} = (-\cos \pi) - (-\cos 0) = (1) - (-1) = 2 \).
   - (A positive value...this region must be above the \( t \)-axis. If you check geometrically, it is.)
   - \( \int_{\pi}^{3\pi/2} \sin(t) \, dt = -\cos(t) \bigg|_{\pi}^{3\pi/2} = (-\cos \frac{3\pi}{2}) - (-\cos \pi) = (0) - (1) = -1 \).
   - (A negative value...this region must be below the \( t \)-axis. Check geometrically. It is.)
4. Force all area regions to be accounted for as positive area (just as \( |\sin(t)| \) would):
   - \( \int_{0}^{3\pi/2} |\sin(t)| \, dt = 2 + 1 = 3 \).

Now, as we go through this example, you might be noticing what is up here. All we are doing is forcing every region of area to have its area be accounted for as positive. There is a general principle here:

Let \( r_1, r_2, \ldots, r_d \) be the \( d \) values on \([a, b]\) where \( f \) takes a value of zero (ie. \( f(r_i) = 0 \), etc.).

Then, \( \int_{a}^{b} |f(t)| \, dt = \left| \int_{r_1}^{a} f(t) \, dt \right| + \left| \int_{r_2}^{r_1} f(t) \, dt \right| + \cdots + \left| \int_{a}^{r_d} f(t) \, dt \right| \).

In other words, the integral of the absolute value function on all of \([a, b]\) is equal to the sum of the absolute values of the integrals along all individual regions of \([a, b]\) where \( f \) is non-zero. In short, that is to say that when computing \( \int_{a}^{b} |f(t)| \, dt \), you count area above the axis as positive and area below the axis as positive.

And, if you were curious, what would a function \(|f(t)|\) look like for a given \( f(t)\)?

A graph of \( f(t) \):

A graph of \( |f(t)| \):

(Note: all area regions become positive!)