How to best deal with power series when determining radius of convergence

During class, Phan demonstrated a general fact about power series. And, that was that every power series centered at \( x = a \) (namely, \( f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots \)) converges on some interval of radius \( R \), where \( R \geq 0 \). That is to say, that for any \( x \) between \( a - R \) and \( a + R \), the power series will converge. Go farther than \( R \) units from \( a \) with your \( x \) value, and the series will not converge (the value of the \((x-a)^n\) terms will simply get too large).

Perhaps the most important takeaway from this demonstration was not the result itself but how the result was gotten. When Phan worked through the result, the method he used worked for a general \( f(x) \)...which is to say the method he used works for ANY power series! What method did he use? [Drum roll, please]

The ratio test!

Any time you want to compute the radius of convergence for the power series, you can use the ratio test. Let me grab an example from your HW 13 to demonstrate this.

An example of the ratio test at work to determine interval of convergence for a power series

Consider #1 from the Additional Problems. We want to determine the radius of convergence for \( \sum_{n=1}^{\infty} \frac{(x-3)^n}{\sqrt{n} \cdot 4^n} \).

We use the ratio test. Let’s first simplify \( \frac{a_{n+1}}{a_n} \), which is \( \frac{(x-3)^{n+1}}{\sqrt{n+1} \cdot 4^{n+1}} \cdot \frac{\sqrt{n} \cdot 4^n}{(x-3)^n} \). Cancelling \( n \) copies of 4 and \( n \) copies of \((x-3)\) located in the numerator and denominator, we get \( \frac{x-3}{4} \cdot \frac{\sqrt{n+1}}{\sqrt{n}} \). Inside this absolute value, the right fraction approaches 1 since the top and bottom are of the same degree (namely, \( n^{1/2} \) from the sqrt) and the limit of terms of equal degree on top of one another is the ratio of the coefficients (in this case, 1).

Hence, this expression limits to \( \frac{x-3}{4} \cdot 1 \), or just \( \frac{x-3}{4} \). And, we know how this plays out with the ratio test. A value less than 1 means absolute convergence. So, \( |x-3|<4 \) is absolute convergence. In other words, we have at a minimum an interval of convergence of \((-1, 7)\)...namely, all points within 4 units of \( x = 3 \).

At this point, the interesting scenario is when the ratio test would be equal to 1. These are the points where \(|x-3|=4\)...namely the two points that are exactly 4 units away from \( x = 3 \). So, we need to test \( x = -1 \) and \( x = 7 \).

Plugging in \( x = -1 \) yields \( \sum_{n=1}^{\infty} \frac{(-4)^n}{\sqrt{n} \cdot 4^n} \) which simplifies to \( \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \). Since \( \frac{1}{\sqrt{n}} \not\to 0 \), this converges by the alternating series test. Plugging in \( x = 7 \), we get \( \sum_{n=1}^{\infty} \frac{4^n}{\sqrt{n} \cdot 4^n} \) which simplifies to \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \) and diverges by the \( p \)-test.

Hence, our interval of convergence is \([-1, 7)\).
Why does the ratio test work so well with power series?

A power series is of the form \( f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + \cdots \).

But, knowing Taylor’s Theorem, we know what these constants are. We know that \( c_k = \frac{f^{(k)}(a)}{k!} \).

The constants on each term involve a factorial. The series could be expressed as \( \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} \cdot (x - a)^n \). Since the terms involve factorials, what test for convergence comes to mind? Ratio Test!