Good old-fashioned numbers are to regular fractions as polynomials are to partial fractions.

There is a huge analogy between regular fractions and partial fractions. Regular fractions relate to regular numbers. Partial fractions relate to polynomials.

First, let \( P(x) \) and \( Q(x) \) be two polynomials (like \( x^2 \), or \( 2x^3 - x + 1 \), or \( -17x^5 + 2x^2 \), or whatever). As long as \( Q(x) \) is not the zero polynomial, we can write these two polynomials as a ratio, namely \( \frac{P(x)}{Q(x)} \). This ratio of polynomial functions should feel a lot like a regular fraction.

Consider a regular fraction, say \( \frac{a}{b} \). We say that this fraction is improper if \(|a| \geq |b|\). Because, in this case, we take out some number of whole \( b \)'s from \( a \) and leave a fractionless whole number with a proper fraction over \( b \) after it. Like \( \frac{17}{5} = (3) + \frac{2}{5} \), or \( \frac{-14}{3} = (-4) + \frac{2}{3} \), or \( \frac{5}{5} = (1) \).

Now, to our ratio of functions, \( \frac{P(x)}{Q(x)} \). We say that this ratio is improper if \( \deg(Q) \geq \deg(P) \). Because, in this case, we can take out some number of whole \( Q \)'s from \( P \) and leave a fractionless polynomial function with a proper fraction after it. Like \( \frac{3x^3 + 5x^2 + 2}{x^2 + 1} = (3x + 5) + \frac{-3x - 3}{x^2 + 1} \). Note the \( 3x + 5 \) portion that results from the long division is a fractionless polynomial function. And, the fraction after is over \( x^2 + 1 \) and proper...namely numerator degree is less than denominator degree.

Another example is \( \frac{2x^2 - 7x + 5}{x^2 - 1} \). This is not proper, since the numerator degree is not less than the denominator degree. This ratio of functions can be divided and simplified. The result is \( 2 + \frac{-7x + 7}{x^2 - 1} \). Again, note, the result of the long division is a fractionless polynomial followed by a proper fraction over the denominator. Completely analogous to good old-fashioned numbers.

So what? What happens when I have a partial fraction integral to compute?

Say you are given an integral of the form \( \int \frac{P(x)}{Q(x)} \, dx \). First, check if the ratio is proper. If the denominator degree is not less than the numerator degree, then long divide. With what results, you can do a partial fraction decomposition on the proper fraction that remains from the division.

A note about the integral of 1 over a linear polynomial

A lot of times with partial fractions, you end up with integrals like \( \left( \int \frac{3}{x - 5} \, dx \right) + \left( \int \frac{-2}{4x - 1} \, dx \right) \). Factoring out constants, we get \( 3 \left( \int \frac{1}{x - 5} \, dx \right) - 2 \left( \int \frac{1}{4x - 1} \, dx \right) \). But, now the gut instinct is probably to substitute...letting \( u = x - 5 \) on the left and \( u = 4x - 1 \) on the right. But, this is cumbersome. We'd have to find \( du \) twice over. It is also confusing, since we'd have two \( u \)'s to negotiate. Errors could crop up here very easily.
A simple alternative is to consider any integral with a linear polynomial in the denominator, namely:

$$\int \frac{1}{mx + b} \, dx$$

where \( m \) and \( b \) are just numbers.

We want this to be something like \( \ln|m x + b| + C \). But, it can’t be just this, because taking the derivative of \( \ln|m x + b| + C \), the chain rule gives us \( \frac{1}{mx + b} \cdot m \). This is \( m \) times what we need. So, if we divide by a factor of \( m \), we’ll have it.

Doing just this we get,

$$\int \frac{1}{mx + b} \, dx = \frac{\ln|m x + b|}{m} + C.$$  

Memorize this. It is gold in partial fraction problems. It saves a ton of substitution.

For example, let’s go back to the situation we started looking at earlier:

$$3 \cdot \left( \int \frac{1}{x - 5} \, dx \right) - 2 \cdot \left( \int \frac{1}{4x - 1} \, dx \right).$$

Using the fact we just derived, we can get right to it:

$$\frac{3 \cdot \ln|x - 5|}{1} + \frac{2 \cdot \ln|4x - 1|}{4} + C.$$  

Problem solved. (This beats the heck out of having to substitute \( u \) into both of the proper fractions we needed to integrate.)