Mon Apr 25 Lecture Commentary

Integrating Polar Functions

When Phan went through the heart area example in class, he included an explanation of how the integration technique he used (dividing the region of area into pie pieces) worked. It was a lot to process, and the integration method he used works for any function in polar form. So, I wanted to take a moment to get the idea down here.

A polar function is of the form \( r(\theta) \). That is to say, for any given \( \theta \), the value of \( r(\theta) \) is the radius from the origin of the point on the graph for that given \( \theta \). Unlike our traditional expressions of functions which can be thought of as having graphs etched horizontally from left to right as \( x \) varies, polar functions are often thought of as being etched radially/circularly in the counter-clockwise manner, from \( \theta = 0 \) to \( \theta = 2\pi \).

Here is an example:

\[
\theta = 5 - 5 \sin \theta
\]

You can imagine drawing that graph by starting at the standard coordinate \((5,0)\) and moving counterclockwise around the perimeter of the figure.

With our traditional functions, we find the area between the graph of the function and the \( x \)-axis. With polar functions, we find the area between the graph of the function and the origin.

With traditional functions, we find the area by breaking the horizontal \( x \)-axis into small rectangles \( dx \) wide and integrating over a horizontal interval of interest, like \([a,b]\). With polar functions, we find the area by breaking the counter-clockwise radial/circular motion into small pie pieces with \( d\theta \) of an angle opening. We then integrate over a radial interval of angles of interest. For example, if we integrated the function above over the interval \([\pi/2, \pi]\), this would give us the area between the graph and the origin in the 2\(^{nd}\) quadrant).

How exactly to integrate polar functions

Let’s consider a generic \( r(\theta) \) on a given interval... let’s say \([\pi/4, 2\pi/3]\). Below is a picture:

We are looking for the area inside \( r(\theta) \) between the dashed lines.
How are we going to integrate this? We'll proceed just like we did with traditional functions and rectangles but with our polar function and pie pieces. Let's get a look at a representative pie piece:

![Pie piece diagram](image)

Now, let's look closer at that representative slice. We ultimately want to develop a method to determine its area.

Area of circle of radius $r(\theta) = \pi (r(\theta))^2$

Area of portion of circle of radius $r(\theta)$ with angle $d\theta = \pi (r(\theta))^2 \cdot \frac{d\theta}{2\pi}$

Note the $\frac{d\theta}{2\pi}$ term is just the fractional portion of the circle we are considering (ie. part over whole). And, also note that when we simplify this area computation out, we can cancel the $\pi$'s and get just $\frac{1}{2} (r(\theta))^2 d\theta$.

Integrating over the region of interest, we get the total area between $r(\theta)$ and the origin on $[\pi/4, 2\pi/3]$ (ie. the area of the region between $r(\theta)$, the origin, and the two dashed lines) to be:

$$\int_{\pi/4}^{2\pi/3} \frac{1}{2} (r(\theta))^2 d\theta.$$ 

Note: this works for any polar function. And, if we wanted to generalize to include an arbitrary area region between any two angles, say $[\theta_1, \theta_2]$, we could say:

$$\int_{\theta_1}^{\theta_2} \frac{1}{2} (r(\theta))^2 d\theta.$$ 

(The formula for finding the area between $r(\theta)$ and the origin between the angles $\theta_1$ and $\theta_2$.)