Fri Mar 4 Lecture Commentary

Working with the identity $\sin^2(\theta)+\cos^2(\theta)=1$

Just a note about where one can see this identity coming from. (There are multiple avenues.) But, ultimately, this can help set up for (and make sense) of the relationship between the hyperbolic trig functions as well.

On one hand, $\sin^2\theta + \cos^2\theta = \frac{\text{opp}^2}{\text{hyp}^2} + \frac{\text{adj}^2}{\text{hyp}^2} = \frac{\text{opp}^2 + \text{adj}^2}{\text{hyp}^2}$. But, no matter what right triangle you are in you can think in terms of Pythagoras to note that $\frac{\text{opp}^2 + \text{adj}^2}{\text{hyp}^2} = \text{hyp}^2$. Substituting in, we get $\sin^2\theta + \cos^2\theta = \frac{\text{hyp}^2}{\text{hyp}^2} = 1$. And, we have it.

But, is there are cheaper way to pull this off? No doubt.

Consider the unit circle, namely all solutions to $x^2 + y^2 = 1$ in the $xy$-plane. We know that points on this circle can be expressed in the form $(x, y) = (\cos \theta, \sin \theta)$. Substitute into the equation. Bam!

In some sense, since the standard trig functions are so intimately related to the circle, we could call them circular trig functions. (We don’t, since if we just say trig functions, we assume the circular trig functions. But, it begs the question if there are hyperbolic trig functions or parabolic trig functions. The answer is yes!)

So, the (circular) trig functions we are most used to can express points on the unit circle. Perhaps the hyperbolic trig functions can express points on the unit hyperbola.

What is the equation of the unit hyperbola? This will dig deep back into your Algebra 2 from HS, but the standard unit hyperbola is $x^2 - y^2 = 1$. (Just switching the + to a − from the circle equation.) Now, let’s build the hyperbolic trig functions by saying points on the unit hyperbola are of the form $(x, y) = (\cosh \theta, \sinh \theta)$.

Substituting $x$ and $y$ into the unit hyperbola equation, we get $\cosh^2\theta - \sinh^2\theta = 1$...precisely the relation that Phan mentions hold. And, for good reason. Hyperbolic trig functions are just ways to express points on a units hyperbola like (circular) trig functions are just ways to express points on a unit circle. It’s no more complicated than that.

And what about the derivatives of the hyperbolic trig functions? Those are funky too.

Consider $\cosh^2\theta - \sinh^2\theta = 1$ and take the derivative of both sides with respect to $\theta$. This will require the chain rule, but we get $2 \cdot \cosh \theta \cdot \frac{d}{d\theta} (\cosh \theta) - 2 \cdot \sinh \theta \cdot \frac{d}{d\theta} (\sinh \theta) = 0$. Dividing 2 from both sides and redistributing the terms, we get:

$$\cosh \theta \cdot \frac{d}{d\theta} (\cosh \theta) = \sinh \theta \cdot \frac{d}{d\theta} (\sinh \theta).$$

This equation has a simple solution. In particular, $\frac{d}{d\theta} (\cosh \theta) = \sinh \theta$ and $\cosh \theta = \frac{d}{d\theta} (\sinh \theta)$. Let your eyes gaze over these three equations and see why it works. You can see, it is completely natural. And, it all stems from a way of expressing points on the unit hyperbola, $x^2 - y^2 = 1$, as $(x, y) = (\cosh \theta, \sinh \theta)$. 