The subtle difference in the computations for $L_N$ and $R_N$.

Let's consider a function $f$ on $[a, b]$.

Whether we are computing $L_N$ or $R_N$, we are estimating the area between the function $f$ and the $x$-axis on the interval $[a, b]$ using rectangles. In both cases, we are dividing the interval from $a$ to $b$ into equal-sized subintervals. So, in other words, the base of each rectangle is of equal width, namely $\frac{b-a}{N}$ (an interval of length $b-a$ broken into $N$ equal pieces). We call this value $\Delta x$ since it is just a small piece of the $x$-axis.

The difference between $L_N$ and $R_N$ is in what endpoint of each subinterval we use to determine the height of each rectangle. In the case of $L_N$, we use the function value at the left endpoint to determine the height of our rectangle. In the case of $R_N$, we use the function value at the right endpoint.

So, let's compute $L_N$. On the first subinterval (ie. $[x_0, x_1]$), we choose the left endpoint $x_0$. Our first rectangle has a height of $f(x_0)$. On $[x_1, x_2]$, the second subinterval, we choose the left endpoint $x_1$. And so on. The last subinterval is $[x_{N-1}, x_N]$. Here, we again choose the left endpoint $x_{N-1}$. The sum of the areas of all of our rectangles will then be $\Delta x \cdot f(x_0) + \Delta x \cdot f(x_1) + \cdots + \Delta x \cdot f(x_{N-1})$. Since $\Delta x$ is constant, we can factor it out, and we get $L_N = \Delta x \cdot \sum_{k=0}^{N-1} f(x_k)$.

To compute $R_N$, we simply choose the right endpoint of each subinterval each time. So, on the first subinterval $[x_0, x_1]$, we choose $x_1$. The height of the first rectangle used in our area estimate will be $f(x_1)$. On the last subinterval $[x_{N-1}, x_N]$, we choose $x_N$. With this, we can sum the areas of all of the rectangles that are a part of our right-endpoint area estimate: $\Delta x \cdot f(x_1) + \Delta x \cdot f(x_2) + \cdots + \Delta x \cdot f(x_N)$. Again, $\Delta x$ is constant, so we factor it out. The result is $R_N = \Delta x \cdot \sum_{k=1}^{N} f(x_k)$.

Let's see these two expressions side-by-side:

$L_N = \Delta x \cdot \sum_{k=0}^{N-1} f(x_k)$ and $R_N = \Delta x \cdot \sum_{k=1}^{N} f(x_k)$.

In effect, the only difference in the expressions for $L_N$ and $R_N$ is that one is a sum from $k = 0$ to $k = N-1$ and the other is a sum from $k = 1$ to $k = N$. The difference is very subtle to notice, but it is ultimately just a consequence of which endpoints on each subinterval are used to determine the heights of each rectangle.
A reminder of a fact about infinite limits

In general, if you are taking the limit \( \lim_{N \to \infty} \frac{A}{B} \), where \( A \) and \( B \) are just polynomials in \( N \), you only need to look at the highest degree terms of \( A \) and \( B \). If the numerator degree is greater, the limit equals \( \infty \). If the denominator degree is greater, the limit equals \( 0 \). If the numerator and denominator degrees are equal, then the limit is the ratio of the coefficients on those highest degree terms.

In particular, there was a case where \( \lim_{N \to \infty} \frac{9(N+1)}{2N} \) was computed. This limit is just \( \lim_{N \to \infty} \frac{9N+9}{2N} \). Noting the numerator and denominator are of the same degree (one in each case), this infinite limit is just the ratio of the leading coefficients, namely \( 9/2 \).

What is a Riemann Sum

If you want to, you could think of a Riemann Sum as just a more flexible version of \( L_N \) or \( R_N \) to approximate the area between a function and the \( x \)-axis using rectangles. In the case of \( L_N \) or \( R_N \), each subinterval is of equal width (ie. the \( \Delta x \) value is constant across all subintervals). In the case of a Riemann Sum, the subintervals can be of any desired width (ie. the \( \Delta x \) values will vary depending on which subinterval is being considered…which is why we use the notation \( \Delta x_i \), because each subinterval can be of different width).

Here is a visualization of an interval from \([a, b]\) subdivided for a Riemann Sum:

Note the first subinterval looks wider than the second subinterval (ie. \( \Delta x_1 > \Delta x_2 \)). This does not occur in the simpler \( L_N \) or \( R_N \) computation, but this is a demonstration of the greater flexibility of the Riemann Sum.

Moreover, when determining which function value on each subinterval will determine the height of the rectangle used on that subinterval to approximate the area between the function and the \( x \)-axis, the Riemann Sum adds flexibility when compared to \( L_N \) or \( R_N \). When computing \( L_N \), you used the collection of sample points \( \{x_0, x_1, \ldots, x_{N-1}\} \) (ie. the left endpoints) and their corresponding function values to dictate the height of each rectangle in your area estimate. For \( R_N \), the collection of sample points was \( \{x_1, x_2, \ldots, x_N\} \) (ie. the right endpoints). For a Riemann Sum, you can use any collection of \( N \) points on \([a,b]\), so long as one point comes from each of the subintervals. You could use all left endpoints or right endpoints. But, you could also use any points from each subinterval. In general, whatever collection of sample points is to be used goes by \( C \) and the points themselves as \( \{c_1, c_2, \ldots, c_N\} \).

To visualize the previously subdivided interval with sample points chosen from anywhere in each subinterval:

Note, \( c_1 \) is in the first subinterval, \( c_2 \) in the second, etc. It appears \( c_2 \) is the right endpoint \([x_1, x_2]\), but this is okay since any point from within that interval is an acceptable choice for that sample point.
Notation for a Riemann Sum computation

The value of the area estimate using widths as the subintervals chosen and heights as the function values at the sample points chosen is \( \Delta x_1 \cdot f(c_1) + \Delta x_2 \cdot f(c_2) + \cdots + \Delta x_N \cdot f(c_N) \) ...the sum of the areas of the \( N \) rectangles used to estimate the area.

In summation notation, this is \( \sum_{k=1}^{N} \Delta x_N \cdot f(c_N) \).

This sum is referred to by \( R(f, P, C) \). Here, \( R \) stands for Riemann Sum, \( f \) is your given function, \( P \) is your partition of subinterval endpoints (ie. \( \{a, x_1, x_2, \ldots, x_{N-1}, b\} \)), and \( C \) is your collection of sample points (ie. \( \{c_1, \ldots, c_N\} \)).

Note, the exact value you get for the Riemann Sum will depend on the function (a taller function will yield a larger area estimate, for example). The value of the Riemann Sum will also depend on the partition (a partition will more endpoints will yield more rectangles and hence a more accurate area estimate, for example). And, the value of the Riemann Sum will additionally depend on the collection of sample points chosen (choosing the point associated with the maximum function value on each subinterval will if anything overestimate the area, whereas choosing the point associated with the minimum function value on each subinterval will if anything underestimate the area).

This is why the expression for the Riemann sum is given as \( R(f, P, C) \). Just as the expression for a function is given as \( f(x) \) because the function value depends on the value of \( x \), the expression for a Riemann sum is given as \( R(f, P, C) \) since the value of the Riemann Sum area estimate depends on the function \( f \), the partition \( P \), and the collection of sample points \( C \).