Fri Apr 15 Lecture Commentary

A way to think about Power Series

First off, note that power series are not on Test 4.

Phan only introduced power series at the very end of class, but I wanted to take this chance to introduce a way of thinking regarding them.

First off, let’s be clear about what a power series is. A power series is a series of the form \( \sum_{n=0}^{\infty} c_n (x-a)^n \), where the \( c_n \)'s are just numbers and \( a \) is just some number. \( x \), the other hand, is a variable.

If we expand out this general form of a power series, it looks like \( c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots \). This looks a lot like a Taylor expansion centered at \( x=a \), and in fact the two ideas are completely related. A power series is nothing more than a Taylor expansion with infinitely many terms. Also note, if we let \( a=0 \) (ie. essentially center our Taylor expansion at \( x=0 \)), we get \( c_0 + c_1x + c_2x^2 + cx^3 + \cdots \). This is basically a polynomial with infinitely many terms (which is a fine way of thinking about power series centered at \( x=0 \)).

If we look at the general form of a power series in summation notation, it looks like \( \sum_{n=0}^{\infty} c_n (x-a)^n \). There are two takeaways here. First, this is a series (a sum with infinitely many terms…it might converge or diverge). Second, this is a function with a variable (the value the sum takes depends on what we plug in for \( x \)).

With this in mind, it is conceivable that for some values of \( x \), the series might have values for its terms that are small enough that the series converge. It is also conceivable that for other values of \( x \), the series might have values for its terms that get large enough that the series does not converge. Determining exactly what values of \( x \) for which a power series converges is of primary importance when studying power series in a calculus course!

A final thought to help develop some instinct regarding what values of \( x \) a power series might converge for

Consider a power series centered about \( x=0 \), namely \( c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots \) (or just \( \sum_{n=0}^{\infty} c_n x^n \)). If we plug in \( x=0 \), all of the terms involving \( x \) will equal zero, and the series will take the value of \( c_0 \)…which is finite, and series converges. If we plugged in \( x=100 \), the terms involving \( x \) are getting gigantic. Unless the \( c_n \)'s are extremely small (and able to multiply by those gigantic numbers to keep the terms of the series small), the sum is likely to diverge (each term in the sum is just so likely to get huge fast). On the other hand, if we plugged in \( x=0.01 \), the terms involving \( x \) are fairly small. Unless the \( c_n \)'s are extremely large (growing faster than the \( x^n \)'s are shrinking), the sum is likely to converge.

In some sense, the closeness of the \( x \) value to the center point of the power series, the more likely the power series is to converge. The farther away an \( x \) value becomes from the center point of the power series, the less likely the power series is to converge.

Phan will probably formalize this idea in the future, but this general principle in fact is universal. Every power series will have a radius of a certain length about the center point of the power series, \( x \) values from within which will allow the power series to converge, and \( x \) values outside of which will force the power series to diverge.