2008 Pro2Serve Tennessee Math Contest

FERMAT I

Answers

1. The solution of \( \sqrt{x} + \sqrt{x-1} = 2 \) is

   a) \( \frac{16}{25} \)  b) \( \frac{25}{16} \)  c) \( \frac{25}{24} \)  d) \( \frac{16}{9} \)  e) none of these

2. An equation of the perpendicular bisector of the line segment joining (1,-3) and (7,5) is

   a) \( 4x+3y = 27 \)  b) \( 3x-4y = 8 \)  c) \( 3x+4y = 16 \)  d) \( 4x-3y = 21 \)  e) \( 3x+4y = 27 \)

3. The remainder when \( x^3 - 4x^2 + 13 \) is divided by \( x-2 \) is

   a) \(-11\)  b) \(11\)  c) \(-5\)  d) \(5\)  e) none of these

4. The 20th prime number is

   a) 61  b) 67  c) 71  d) 73  e) 79

5. The solution of the equation \( \ln x + \ln(x-1) = 1 \) is\( \sqrt{e} \)

   a) \( \frac{1}{2} \left( 1 - \sqrt{1-4e} \right) \)  b) \( \frac{1}{2} \left( 1 + \sqrt{1-4e} \right) \)  c) \( \frac{1}{2} \left( 1 - \sqrt{1+4e} \right) \)
   d) \( \frac{1}{2} \left( 1 + \sqrt{1+4e} \right) \)  e) \( \frac{1}{2} \left( 1 \pm \sqrt{1+4e} \right) \)

6. If \( X \) and \( Y \) are the midpoints of the sides \( AB \) and \( AC \), respectively, of a triangle and the length of the line segment joining \( X \) and \( Y \) is 12, what is the length of side \( BC \) of the triangle?

   a) 6  b) 9  c) 12  d) 24  e) 36
7. The determinant of the matrix \( A = \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix} \) is

a) 1 \quad b) 23 \quad c) 7 \quad d) -1 \quad e) none of these

8. From a group of 9 different books, 5 books are to be selected and arranged on a shelf. How many arrangements are possible?

a) 15,120 \quad b) 126 \quad c) 13,440 \quad d) 3,024 \quad e) none of these

9. The real values of \( x \) for which \( f(x) = -x^2 + (\sqrt{2} - \sqrt{3}) x + \sqrt{6} \) is positive are

a) \( x < -\sqrt{3} \) or \( x > \sqrt{2} \)

b) \( x \leq \sqrt{2} \)  

c) \( -\sqrt{3} \leq x \leq \sqrt{2} \)

d) \( -\sqrt{3} < x < \sqrt{2} \)

e) \( x \geq -\sqrt{3} \)

10. Three vertices of a rectangle are (9,3), (5,9), and (-7,1). The fourth vertex is

a) (-2, -4) \quad b) (-3, -6) \quad c) (-2, -5) \quad d) (-1, -3) \quad e) none of these

11. The Smiths' den is square and the Chases' den is rectangular (not square). The Chases' den measure 3m less on one side than the Smiths', 2m more on the adjacent side, and has 300m\(^2\) of floor space. What is the difference in the areas of the two dens?

a) 6 \quad b) 11 \quad c) 18 \quad d) 24 \quad e) 36

12. The number of real roots of \( t^4 + 3t^3 + 6t^2 - 2t - 60 = 0 \) is

a) 0 \quad b) 1 \quad c) 2 \quad d) 3 \quad e) 4

13. How many positive even integers less than 1,000 can be written in base 10 notation using the digits 2,3,4,5?

a) 120 \quad b) 24 \quad c) 48 \quad d) 42 \quad e) 499
14. Which of the following inequalities is always true for positive numbers \( a \) and \( b \)?

a) \( 2\sqrt{a+b} \leq \sqrt{a} + \sqrt{b} \)  

b) \( 2\sqrt{ab} \leq a + b \)  

c) \( \sqrt{ab} \leq \sqrt{a} + \sqrt{b} \)

d) \( \sqrt{a+b} \leq \sqrt{a} + \sqrt{b} \)  

e) none of these

15. The linear system of equations \[
\begin{align*}
x + y + 2z &= a \\
x + z &= b \\
2x + y + 3z &= c
\end{align*}
\] is consistent (has a solution) for constants \( a, b, \) and \( c \) only if

a) \( a = b + c \)  

b) \( b = a + c \)  

c) \( c = a + b \)  

d) \( a = b - c \)  

e) \( b = a - c \)

16. Let \( \alpha \) and \( \beta \) be the acute angles of a right triangle. If \( \sin^2(2\beta) - 2\cos(2\beta) = 0 \), then \( \sin \alpha \) is

a) \( \frac{\sqrt{2}}{2} \)  

b) \( \frac{\sqrt{2}}{2} \)

c) \( \frac{\sqrt{8}}{2} \)  

d) \( \frac{\sqrt{8}}{4} \)  

e) 0

17. Let \( P \) be a regular polygon with 20 sides. The number of diagonals of \( P \) which do not pass through its center is

a) 170  

b) 150  

c) 340  

d) 320  

e) 160

18. For which value of \( k \) do the graphs of \( y = x + k \) and \( x = y^2 \) intersect in exactly one point?

a) \( \frac{\sqrt{2}}{4} \)  

b) \( \frac{1}{2} \)  

c) \( \frac{1}{4} \)  

d) \( \frac{\sqrt{2}}{8} \)  

e) none of these

19. Let \( f(x) = \frac{1+x}{1-x} \). Then the product \( f(f(x))f(\frac{1}{f(x)}) \) is

a) \( \frac{-1}{x^2} \)  

b) 1  

c) \( \frac{1}{x^2} \)  

d) \( x^3 \)  

e) not defined
20. Mrs. Smith bought 40 lbs of strawberries at $3 per pound from Mr. Farmer but 10% of the strawberries were found to be rotten at the time of delivery. When Mrs. Smith sells the non-rotten strawberries, what should the selling price per pound be so that her profit is 20%?

a) \$3.50  \hspace{1cm} (b) \$4.00  \hspace{1cm} c) \$4.50  \hspace{1cm} d) \$5.00  \hspace{1cm} e) none of these

21. Let \( x, y, \) and \( z \) be distinct real numbers. The expression \( \frac{x^2 - 1}{(x-y)(x-z)} + \frac{y^2 - 1}{(y-x)(y-z)} + \frac{z^2 - 1}{(z-x)(z-y)} \) is equal to

a) \( x + y + z \)  \hspace{1cm} b) \( xyz \)  \hspace{1cm} c) 0  \hspace{1cm} d) \( x(y-z) \)  \hspace{1cm} e) \( 1 \)

22. \( \sqrt{((1000003)^3 - (999997)^3 - (18)(1000000)^2} - 27 \) is equal to

a) 1000002  \hspace{1cm} b) 999998  \hspace{1cm} c) 500000  \hspace{1cm} d) 27  \hspace{1cm} e) none of these

23. Let \( A \) be a square with side length 1 and let \( S \) be the square inscribed in \( A \) formed by joining the midpoints of the sides of \( A \). The area of a circle inscribed in \( S \) is

a) \( \frac{\pi}{2} \)  \hspace{1cm} b) \( \frac{\sqrt{2}}{2} \pi \)  \hspace{1cm} c) \( \frac{\pi}{8} \)  \hspace{1cm} d) \( \frac{\pi}{4} \)  \hspace{1cm} e) none of these

24. Let \( x, y, \) and \( z \) be real numbers such that \( x^2 + y^2 + z^2 + 2x + 4y + 6z + 14 = 0 \). Then the product \( xyz \) is equal to

a) 0  \hspace{1cm} b) 12  \hspace{1cm} c) -6  \hspace{1cm} d) 6  \hspace{1cm} e) 9

25. The solution set for \( |x + 7| - 4 \geq -1 \) is

a) \( \{ x \mid -10 < x < -4 \} \)  \hspace{1cm} b) \( \{ x \mid x < -10 \text{ or } x > -4 \} \)  \hspace{1cm} c) \( \{ x \mid x \leq -10 \text{ and } x \geq -4 \} \)  \hspace{1cm} d) \( \{ x \mid x < -10 \text{ and } x > -4 \} \)  \hspace{1cm} e) \( \{ x \mid x \leq -10 \text{ or } x \geq -4 \} \)