2012 FERMAT II TEST

INSTRUCTIONS:

• Each problem has the same credit.

• For each problem, give as complete a solution as you can.

• You will be graded on both the correctness of your answer and the quality of your explanation / justification.

NAME: ____________________________________________

SCHOOL: __________________________________________

The table below is strictly for the graders’ use.

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1. Given an integer $n \geq 3$, show that $n^3 + 8$ is divisible by at least two distinct primes.
2. Suppose $A, B, C, D$ are points in the plane such that no three of them are collinear. If $D$ is in the interior of $\triangle ABC$, show that

$$|AD| + |BD| + |CD| < |AB| + |BC| + |CA|.$$  

(Note: $|XY|$ denotes the length of the line-segment $XY$.)
3. Let \( x, y, z, w \) be positive real numbers such that \( x + y + z = w \).
Show that
\[
\frac{(w - x)(w - y)(w - z)}{(w + x)(w + y)(w + z)} \leq \frac{1}{8}.
\]
4. Suppose $A, B, C$ are vertices of a triangle and $D$ is a point on the edge $BC$. Let $L$ be the line that contains $A$ and bisects $\angle CAB$. Suppose there is a point $E$ on $L$ such that upon drawing the line segments $EC$ and $DE$, we have $\angle AEC = \angle ABC$ and $\angle CDE = 90^\circ$. Then, show that $|BD| = |CD|$. (Note: $|XY|$ denotes the length of the line-segment $XY$.)
5. Determine all integer triples \((x, y, z)\) such that \(1 \leq x \leq y \leq z\) and
\[
x + y + z + xy + yz + xz = xyz - 1.
\]
6. Suppose $P$ is an 11-sided regular polygon and $S$ is the set of all lines that contain two distinct vertices of $P$. If three lines are randomly chosen from $S$, what is the probability that the chosen lines contain a pair of parallel lines?
7. Let \( x, y, z \) be positive integers with greatest common divisor 1. If

\[
\frac{1}{x} + \frac{1}{y} = \frac{1}{z},
\]

then show that \( \sqrt{x + y} \) is an integer.
8. If $m$ and $n$ are positive integers, then show that

$$\frac{m}{\sqrt{n}} + \frac{m}{\sqrt[n]{n}} \neq 1.$$