2005 FERMAT II TEST

1) Let \( k \) be a positive integer. Show that the sum of the first \( 2k - 1 \) odd positive integers can be expressed as a sum of \( 2k - 1 \) consecutive positive integers.

2) A square \( S \) is inscribed in a triangle \( T \) (so that the vertices of \( S \) are on the sides of \( T \)). Show that the area of \( T \) is at least twice the area of \( S \).

3) Show that for all positive real numbers \( X, Y, \) and \( \alpha \) the following inequality holds.
\[
X \sin^2 \alpha Y \cos^2 \alpha < X + Y
\]

4) Let \( f(X) \) be a polynomial with integral coefficients such that \( f(2004) = f(2005) = 5 \). Show that \( f(n) \neq 50 \) for all integers \( n \).

5) Suppose a hexagon with sides of alternating lengths 1 and 2 is inscribed in a circle of radius \( r \). Find \( r \).

6) Show that there are only finitely many integers \( x, y \) such that \( x^2 - xy + y^2 = 2005 \).

7) Compute \( a_1 + a_2 + \cdots + a_{2005} \) where \( a_n := \frac{2n + 1 + \sqrt{n^2 + n}}{\sqrt{n} + \sqrt{n + 1}} \) for all positive integers \( n \).

8) The top of a cube of side-length 1 is a square whose sides have mid-points \( A, B, C, D \) as shown in the figure below. The mid-points of the sides of the bottom square are similarly labeled \( A', B', C', D' \).

What is the side-length of the square whose vertices are the mid-points of \( AB', BC', CD', DA' \)?