2004 Fermat II Test

1) What is the maximum number of points in the plane such that the distance between any two of them is the same?

2) Is the product of $2002^{2004}$ and $2004^{2002}$ greater than $2003^{4006}$?


4) Do there exist positive integers $x, y, a, b$ such that $x^2 + y = a^2$ and $y^2 + x = b^2$?

5) Let $a_1, \cdots, a_{2004}$ be real numbers such that $a_1 = 2004$ and $a_1 + \cdots + a_n = n^2 a_n$ for all $1 \leq n \leq 2004$. Compute $a_{2004}$.

6) In the figure below, the 8 smaller circles shown are of the same radius whereas the larger circle has radius 1. Each of the smaller circles touches both of its neighbors and the larger circle. What is the radius of each of the smaller circles?

7) Let $a, b$ be distinct integers. Prove that the polynomial $(X - a)^2 (X - b)^2 + 1$ cannot be expressed as a product of two quadratic polynomials having integer-coefficients.
8) Among all rational numbers strictly greater than $\frac{13}{1309}$ but strictly less than $\frac{11}{1107}$ find one which has the least positive denominator.

Alternate 2004 Fermat II Test

1) Every point on the plane is colored either green or red (but not both). Show that there exist at least two points of the same color that are at distance 1 from each other.

2) Let $a := 1 \times 2 \times 3 \times \cdots \times 998 \times 999$ and $b := 500^{999}$. Which of the integers $a$ and $b$ is the lesser?

3) Let $A$, $B$, $C$ be the vertices of a triangle such that the side $AB$ measures 10, the side $BC$ measures 21 and the side $AC$ measures 17. Let $P$, $Q$, $R$ be the vertices of a square inscribed in the triangle $ABC$ so that the side $PQ$ of the square lies along $BC$, vertex $R$ is on the side $AC$ and vertex $S$ is on the side $AB$. Find the length of $PQ$.

4) Do there exist positive integers $x$, $y$ such that $x^3 + 2x^2 + x + 1 = y^3$?

5) Let $a_1, \cdots, a_{2004}$ be a sequence of non-negative real numbers such that $2\sqrt{a_n} \geq a_{n+1} + 1$ for all $1 \leq n \leq 2003$ and $2\sqrt{a_{2004}} \geq a_1 + 1$. Compute $a_{2004}$. 
6) Let A, B, C be the vertices of an equilateral triangle of side length 1. Find the radii of the two circles shown in the figure below.

7) Let \( f(x) \) be a non-constant, integer-coefficient polynomial. Prove that the product \( f(x)f(x+1) \) cannot be equal to the polynomial
\[ (x-1)(x-2)\cdots(x-2004)+1. \]

8) Let \( b_1, \ldots, b_n \) be \( n \) real numbers such that each of them is 2 or greater and
\[ \frac{1}{b_1^2} + \cdots + \frac{1}{b_n^2} \geq 1. \]