2003 Fermat II Test

1) Show that the area of a regular 8-gon is equal to the product of its longest diagonal and its shortest diagonal.

2) Determine all positive integers $n$ such that $n^2 = 2^n$.

3) The diagonals AC and BD of a square ABCD meet at O. The bisector of $\angle BAO$ meets BO, BC at N, P respectively. The length of NO is 24. Find the length of PC.

4) Show that $\sqrt[3]{5} + 2 - \sqrt[3]{5} - 2 = 1$. (No credit for verification using Calculators.)

5) There are 100 people standing in a row. Heights of any two neighbors differ by 1 inch. Show that it is possible to rearrange these people in two groups such that the sum of the heights of members in either group is the same.

6) 400 students took a test consisting of 50 true-false questions. No partial credit was given on any question. Each of the questions was answered correctly by 300 or more students. Show that there was at least one student who answered 38 or more questions correctly.

7) Determine all pairs $(p, q)$ of prime numbers such that the equation $x^4 - px^3 + q = 0$ has an integer solution.

8) Show that the equation $x^6 + cx^4 + cx^2 + 1 = 0$, where $c$ is a real number other than -1, has either no solution in rational numbers or exactly four distinct solutions in rational numbers.