Bergman-type reproducing kernels and invariant subspaces

(and complete Nevanlinna-Pick kernels also)

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Disclaimer: To keep matters simple I have modified many theorems and definitions a little. While I have tried to tell the truth, it should be pointed out that often in the original references a larger generality is achieved.
Spaces of analytic functions on the open unit disc $\mathbb{D}$:

$$f(z) = \sum_{n \geq 0} \hat{f}(n) z^n$$

$H^2$, \quad $\|f\|^2_{H^2} = \sum_{n \geq 0} |\hat{f}(n)|^2$

$L^2_a = \{f \in Hol(\mathbb{D}) : \int_{\mathbb{D}} |f|^2 \frac{dA}{\pi} < \infty\},$

$$\|f\|^2_{L^2_a} = \int_{\mathbb{D}} |f|^2 \frac{dA}{\pi} = \sum_{n \geq 0} \frac{|\hat{f}(n)|^2}{n + 1}$$

$D = \{f \in Hol(\mathbb{D}) : f' \in L^2_a\},$

$$\|f\|^2_D = \|f\|^2_{H^2} + \|f'\|^2_{L^2_a} = \sum_{n \geq 0} (n + 1)|\hat{f}(n)|^2$$

$$D \subseteq H^2 \subseteq L^2_a$$
\[ \zeta(z) = z \]

\((M_\zeta, \mathcal{H}) \quad M_\zeta f = \zeta f, \; f \in \mathcal{H}. \]

\(\mathcal{M} \in \text{Lat}(M_\zeta, \mathcal{H}), \) if \(M_\zeta f \in \mathcal{M}\) for all \(f \in \mathcal{M}\)

If \(S \subseteq \mathcal{H}, \) then
\([S] = \text{smallest } \mathcal{M} \in \text{Lat}(M_\zeta, \mathcal{H}) \text{ containing } S.\]

\(\text{ind } \mathcal{M} = \dim \mathcal{M} \ominus \zeta \mathcal{M} = \dim \mathcal{M} \cap (\zeta \mathcal{M})_\perp\)

Note: \(\text{ind } [S] \leq \text{card } S\)

\(G\) is the \textit{extremal function} for \(\mathcal{M}, \) if \(G\) is the solution to
\[
\sup \{ \text{Re } g^{(k)}(0) : g \in \mathcal{M}, \|g\| = 1 \},
\]
where \(k\) is the smallest integer such that the sup is \(> 0.\)

If \(\mathcal{M} \neq (0),\) then an extremal function \(G\) exists, is unique, and \(G \in \mathcal{M} \ominus \zeta \mathcal{M}\)
Thm 1 (\(H^2\) - Beurling’s theorem). If 
\(\mathcal{M} \in \text{Lat}(M_\zeta, H^2), \mathcal{M} \neq (0),\) then

\[\text{ind}\mathcal{M} = 1.\]

If \(G\) is extremal for \(\mathcal{M}\), then \(G\) is inner and \(\mathcal{M} = [G] = GH^2\), so

\[\frac{\mathcal{M}}{G} = H^2\] isometrically,

i.e. \(\|f_G\| = \|f\|\) for all \(f \in \mathcal{M}\).
Thm 2 (D - Ri, Sundberg, Aleman). If $\mathcal{M} \in \text{Lat}(M_\zeta, D), \mathcal{M} \neq (0)$, then $\text{ind}\mathcal{M} = 1$.

If $G$ is extremal for $\mathcal{M}$, then $\mathcal{M} = [G]$.

If $\mathcal{M}, \mathcal{N} \in \text{Lat}(M_\zeta, D)$,

$$(0) \neq \mathcal{N} \subseteq \mathcal{M} \subseteq D,$$

if $G_\mathcal{N}, G_\mathcal{M}$ are the extremal functions for $\mathcal{N}$ and $\mathcal{M}$, then one has the contractive inclusions

$$D \subseteq \frac{\mathcal{M}}{G_\mathcal{M}} \subseteq \frac{\mathcal{N}}{G_\mathcal{N}} \subseteq H^2,$$

i.e. $\|G_{\mathcal{M}}f\| \leq \|f\|$ for all $f \in D$,

$\|\frac{G_\mathcal{N}}{G_{\mathcal{M}}}f\| \leq \|f\|$ for all $f \in \mathcal{M}$, and

$\|\frac{f}{G_\mathcal{N}}\|_{H^2} \leq \|f\|$ for all $f \in \mathcal{N}$.
Fact (ABFP): In $L^2_a$ there are invariant subspaces of arbitrary index.

**Thm 3.** Hedenmalm, ARS, H-Jacobsson-Shimorin

If $\mathcal{M} \in \text{Lat}(M_\zeta, D)$, $\mathcal{M} \neq (0)$, then

$$\mathcal{M} = [\mathcal{M} \ominus \zeta \mathcal{M}].$$

If $\mathcal{M}, \mathcal{N} \in \text{Lat}(M_\zeta, L^2_a)$, $\text{ind } \mathcal{M} = \text{ind } \mathcal{N} = 1$,

$$(0) \neq \mathcal{N} \subseteq \mathcal{M} \subseteq L^2_a,$$

if $G_\mathcal{N}, G_\mathcal{M}$ are the extremal functions for $\mathcal{N}$ and $\mathcal{M}$, then one has the contractive inclusions

$$H^2 \subseteq \frac{\mathcal{N}}{G_\mathcal{N}} \subseteq \frac{\mathcal{M}}{G_\mathcal{M}} \subseteq L^2_a,$$

i.e. $\|G_\mathcal{N}f\| \leq \|f\|_{H^2}$ for all $f \in H^2$,

$$\left\| \frac{f}{G_\mathcal{M}} \right\| \leq \|f\| \text{ for all } f \in \mathcal{M}, \text{ and}$$

$$\left\| \frac{G_\mathcal{M}}{G_\mathcal{N}} f \right\| \leq \|f\| \text{ for all } f \in \mathcal{N}.$$
Cor 4. If $f \in L^2_\alpha$, $f \neq 0$, then $f$ has an $L^2_\alpha$-inner-outer factorization, i.e.

$$f = FG,$$

where $G$ is extremal for $[f]$ and $F \in L^2_\alpha$ is cyclic, $[F] = L^2_\alpha$. 
$\mathcal{H} = \mathcal{H}(k)$ has reproducing kernel $k_\lambda(z)$:

$$f(\lambda) = \langle f, k_\lambda \rangle$$

for all $f \in \mathcal{H}$.

$k_\lambda(z) \gg 0$ positive definite, i.e.

$$\forall a_1, \ldots, a_n \sum_{i,j} a_i \overline{a}_j k_\lambda_i(\lambda_j) \geq 0,$$

because $\| \sum_i a_i k_\lambda_i \|^2 = \sum_{i,j} a_i \overline{a}_j k_\lambda_i(\lambda_j)$.

If $\{e_n\}$ is an orthonormal basis for $\mathcal{H}(k)$, then

$$k_\lambda(z) = \sum e_n(\lambda) e_n(z).$$

**Fact:** If $u_\lambda(z) \gg 0$ is analytic in $z$, then $\exists u_n \in Hol(\mathbb{D})$ such that

$$u_\lambda(z) = \sum u_n(\lambda) u_n(z).$$
\[ \mathcal{H}(k^1) := \frac{\mathcal{N}}{G_\mathcal{N}} \subseteq \frac{\mathcal{M}}{G_\mathcal{M}} =: \mathcal{H}(k^2) \]

Abstract nonsense:

Set \( \Vert \frac{f}{G_\mathcal{M}} \Vert_\mathcal{M} = \Vert f \Vert, f \in \mathcal{M}, \Vert \frac{f}{G_\mathcal{N}} \Vert_\mathcal{N} = \Vert f \Vert, f \in \mathcal{N} \).

Then the contractive inequality
\[ \Vert \frac{G_\mathcal{M}}{G_\mathcal{N}} f \Vert \leq \Vert f \Vert \] for all \( f \in \mathcal{N} \)
is equivalent to
\[ \Vert g \Vert_\mathcal{M} \leq \Vert g \Vert_\mathcal{N} \] for all \( g \in \frac{\mathcal{N}}{G_\mathcal{N}} \).

If \( g \in \frac{\mathcal{N}}{G_\mathcal{N}} \), then \( g = \frac{f}{G_\mathcal{N}} \) for some \( f \in \mathcal{N} \), so
\[
\Vert g \Vert_\mathcal{M} = \Vert \frac{f}{G_\mathcal{N}} \Vert_\mathcal{M} = \Vert \frac{G_\mathcal{M}}{G_\mathcal{N}} \frac{f}{G_\mathcal{M}} \Vert_\mathcal{M} = \Vert \frac{G_\mathcal{M}}{G_\mathcal{N}} f \Vert \\
\leq \Vert f \Vert = \Vert \frac{f}{G_\mathcal{N}} \Vert_\mathcal{N} = \Vert g \Vert_\mathcal{N}
\]
\[ \mathcal{M}, \mathcal{N} \subseteq \mathcal{H}(k) \]

\[ \mathcal{H}(k^1) := \frac{\mathcal{N}}{G_{\mathcal{N}}} \subseteq \frac{\mathcal{M}}{G_{\mathcal{M}}} =: \mathcal{H}(k^2) \]

reproducing kernels: \( \langle f, k_\lambda \rangle = f(\lambda) \)

\[ \mathcal{H}(k^1) = k_\lambda \]

\[ \mathcal{M} = P_{\mathcal{M}}k_\lambda \]

\[ \frac{\mathcal{M}}{G_{\mathcal{M}}} = \frac{P_{\mathcal{M}}k_\lambda(z)}{G_{\mathcal{M}}(\lambda)G_{\mathcal{M}}(z)} \]
Thm 5 (Aronszajn). $\mathcal{H}(k^1) \subseteq \mathcal{H}(k^2)$ contractively, if and only if

$$k^1 \ll k^2,$$

i.e. $k^2 - k^1 >> 0$.

Proof: Define $T : \mathcal{H}(k^2) \rightarrow \mathcal{H}(k^1)$ by

$$T \left( \sum a_i k^2_{\lambda_i} \right) = \sum a_i k^1_{\lambda_i},$$

then $T$ is a contraction if $k^1 \ll k^2$, and $T^* : \mathcal{H}(k^1) \rightarrow \mathcal{H}(k^2)$ is the inclusion map. ■
We shall get:

\[
\frac{k^2}{k^1} \gg 0.
\]

**Lemma 6.** If

\[
\frac{k^2}{k^1} \gg 0
\]

and if \( k_0^1(z) = k_0^2(z) = 1 \), then

\[
k^2 - k^1 \gg 0.
\]

Proof: Since \( \frac{k_0^2(z)}{k_0^1(z)} = 1 \) we have

\[
\frac{k^2}{k^1} - 1 \gg 0,
\]

hence by the Schur product theorem

\[
k^2 - k^1 = k^1 \left( \frac{k^2}{k^1} - 1 \right) \gg 0.
\]
Note: In our situation $k_0^1(z) = k_0^2(z) = 1$ for all $z \in \mathbb{D}$:

Say $\mathcal{H}(k) := \frac{M}{G_M}$, where $\text{ind } M = 1$ and $G_M \in M \ominus \zeta M$, $\|G_M\| = 1$.

Then, for $f \in M$ we have

$$\langle f, G_M \rangle = \frac{f}{G_M}(0)$$

(= a property of extremal functions), hence

$$\langle \frac{f}{G_M}, 1 \rangle_M = \frac{f}{G_M}(0),$$

and so

$$\langle g, 1 \rangle_M = g(0) \text{ for all } g \in \frac{M}{G_M}.$$ 

Thus

$$1 = k_0.$$
Interpretation: dilation theorem

**Thm 7 (Agler).** If 

\[
\frac{k^2}{k^1} \gg 0,
\]

then 

\((M_\zeta, \mathcal{H}(k^2))\) dilates to \((M_\zeta^{(\infty)}, \bigoplus \mathcal{H}(k^1))\),

i.e. \(\exists M \in \text{Lat} (M_\zeta^{(\infty)*}, \bigoplus \mathcal{H}(k^1))\) such that 

\((M_\zeta^*, \mathcal{H}(k^2))\) is u.e. \((M_\zeta^{(\infty)*}, \bigoplus \mathcal{H}(k^1)) | M\).

proof omitted in talk
Proof: Write \( u_\lambda(z) = \frac{k^2_\lambda(z)}{k^1_\lambda(z)} \), then
\[
u_\lambda(z) = \sum_n u_n(z)\overline{u_n(\lambda)}, \text{ hence}\]
\[
k^2_\lambda(z) = \sum_n u_n(z)\overline{u_n(\lambda)}k^1_\lambda(z).
\]

Thus, if we set
\[M^*_n : \mathcal{H}(k^2) \rightarrow \mathcal{H}(k^1), M^*_n k^2_\lambda = \overline{u_n(\lambda)}k^1_\lambda,
\]
one checks that each \( M^*_n \) extends to be bounded and
\[
\sum_n ||M^*_n f||^2_1 = ||f||^2_2,
\]
whenever \( f \) is a finite linear combination of reproducing kernels. Thus, the map
\[V : \mathcal{H}(k^2) \rightarrow \bigoplus \mathcal{H}(k^1) \quad Vf = \{M^*_n f\}_n\]
is an isometry. Set \( \mathcal{M} = \text{ran}V \), then the Theorem follows, because
\[V(M^*_\zeta, \mathcal{H}(k^2)) = (M^*_\zeta^{(\infty)} , \bigoplus \mathcal{H}(k^1))V.\]
**Defn 8.** Let \( k \) be a reproducing kernel for \( \mathcal{H}(k) \) with \( k_0(z) = 1 \).

(a) \( k \) is a complete NP kernel, if \( \exists u \gg 0 \) such that

\[
k_\lambda(z) = \frac{1}{1 - u_\lambda(z)}.
\]

(b) \( k \) is a Bergman-type kernel, if \( \exists u \gg 0 \) with \( u_0(z) = 0 \) and \( \exists \rho \in H^\infty, \text{outer}, 1 \leq |\rho|^2 \leq 2, \varphi = \zeta \rho, \)

\[
k_\lambda(z) = \frac{1}{1 - \varphi(\lambda)\varphi(z)(1 - u_\lambda(z))},
\]

and \( ||k_z|| \to \infty \) as \( |z| \to 1 \).

Examples:

\( \mathcal{H}^2: \) \( k_\lambda(z) = \frac{1}{1 - \lambda z} \) is both.

\( \mathcal{D}: \) \( k_\lambda(z) = \frac{1}{\lambda z} \log \frac{1}{1 - \lambda z} \) is a complete NP kernel (proof omitted).

\( \mathcal{L}_a^2: \) \( k_\lambda(z) = \frac{1}{(1 - \lambda z)^2} = \frac{1}{1 - 2\lambda z(1 - \frac{1}{2}\lambda z)} \) is a Bergman-type kernel, \( \varphi(z) = \sqrt{2}z, u_\lambda(z) = \frac{1}{2}\lambda z. \)
\[ k^\beta_{\lambda}(z) = \frac{1}{(1-\lambda z)^\beta}, \quad 1 \leq \beta \leq 2 \] is a Bergman-type kernel.

\[ \frac{1}{k^\beta_{\lambda}(z)} = (1 - \lambda z)^\beta = 1 - \beta \lambda z + \sum_{n \geq 2} c_n (\lambda z)^n, \]

where \( c_n \geq 0 \) for all \( n \geq 2 \), since \( 1 \leq \beta \leq 2 \).

Fact: If \( k \) is a Bergman-type kernel, then

\[ H^2 \subseteq \mathcal{H}(k) \subseteq L_a^2 \]

contractively.
Thm 9 (Shimorin). If $\mathcal{M} \in \text{Lat} \ (M_\zeta, D)$, $\mathcal{M} \neq (0)$, and if $G$ is the extremal function for $\mathcal{M}$, then

$$\frac{P_{\mathcal{M}k\lambda}(z)}{G(\lambda)G(z)}$$

is a complete NP kernel.

Thm 10 (McCullough-Ri). If $\mathcal{M} \in \text{Lat} \ (M_\zeta, L^2_a)$, $\text{ind} \ \mathcal{M} = 1$, and if $G$ is the extremal function for $\mathcal{M}$, then

$$\frac{P_{\mathcal{M}k\lambda}(z)}{G(\lambda)G(z)}$$

is a Bergman-type kernel.

Fact 1: $\exists$ a complete NP kernel $k$ and invariant subspace $\mathcal{M}$ of $(M_\zeta, H(k))$ such that $\frac{P_{\mathcal{M}k\lambda}(z)}{G(\lambda)G(z)}$ is not a complete NP kernel.

Fact 2: If $k$ is a Bergman-type kernel, if $\mathcal{M}$ is a zero-based invariant subspace of $(M_\zeta, H(k))$, then $\frac{P_{\mathcal{M}k\lambda}(z)}{G(\lambda)G(z)}$ is a Bergman-type kernel.
Operator identities:

\[ u_\lambda(z) \gg 0 \text{ so } u_\lambda(z) = \sum u_n(z)u_n(\lambda). \]

\( k \) is an NP kernel, iff

\[ (1 - \sum u_n(z)u_n(\lambda))k_\lambda(z) = 1, \]

iff

\[ (I - \sum M_{un}M_{un}^*)k_\lambda = k_\lambda(0) = Q_0k_\lambda, \]

where \( Q_0f = f(0) \). Thus

\[ I - \sum M_{un}M_{un}^* = Q_0, \]

or

\[ \sum M_{un}M_{un}^* = I - Q_0 = P_0. \]
$k$ is a Bergman-type kernel, iff

$$(1 - \varphi(\lambda)\varphi(z)(1 - u_\lambda(z))) k_\lambda = k_\lambda(0),$$

so

$$\frac{k_\lambda - k_\lambda(0)}{\varphi(\lambda)\varphi(z)} + u_\lambda(z)k_\lambda = k_\lambda$$

$$\frac{P_0 k_\lambda}{\varphi(\lambda)\varphi(z)} + \sum u_n(z)\overline{u_n(\lambda)}k_\lambda = k_\lambda$$

$$\frac{M_1 P_0 M_1^*}{\varphi} \frac{\varphi}{\varphi} + \sum M_{un} M_{un}^* = I$$
Thm 11 (McCullough–Trent). If $k$ is a complete NP kernel, and if $\mathcal{M}$ is a multiplier invariant subspace of $\mathcal{H}(k)$, then

$$\frac{P_\mathcal{M}k_\lambda(z)}{k_\lambda(z)} >> 0.$$ 

Hence, if $G$ is extremal for $\mathcal{M}$, then with

$$k_\lambda^1(z) = \frac{P_\mathcal{M}k_\lambda}{G(\lambda)G(z)},$$

we have

$$\frac{k_\lambda^1}{k} >> 0$$

and we obtain the contractive inclusion

$$\mathcal{H}(k) \subseteq \frac{\mathcal{M}}{G}.$$
Proof: Set $Q(A) = \sum M_{u_n}AM_{u_n}^*$, so
\[ I - Q(I) = Q_0. \]
Also
\[ P_MQ(P_M)P_M = Q(P_M), \]
because $Q(P_M)P_M \perp = 0$, $P_M \perp Q(P_M) = 0$.

Set
\[ T = P_M - Q(P_M) \]
\[ = P_M(I - Q(P_M))P_M = P_M(I - Q(I - P_M \perp))P_M \]
\[ = P_M(I - Q(I)) + Q(P_M \perp))P_M \]
\[ = P_MQ_0P_M + P_MQ(P_M \perp)P_M \geq 0 \]

Thus,
\[ 0 << \langle Tk_\lambda, k_z \rangle = P_Mk_\lambda(z) - \sum \langle M_{u_n}P_MM_{u_n}^*k_\lambda, k_z \rangle \]
\[ = P_Mk_\lambda(z) - u_\lambda(z)P_Mk_\lambda(z) \]
\[ = (1 - u_\lambda(z))P_Mk_\lambda(z) = \frac{P_Mk_\lambda(z)}{k_\lambda(z)} \]
Thm 12 (McCullough-Ri). If $k$ is a Bergman-type kernel, if $\mathcal{M} \in \text{Lat}(M_\zeta, \mathcal{H}(k))$, ind $\mathcal{M} = 1$, if $G$ is the extremal function for $\mathcal{M}$, then

$$\frac{P_Mk_\lambda(z)}{G(\lambda)G(z)} = (1 - l_\lambda(z))k_\lambda(z)$$

for some $l_\lambda(z) >> 0$.

Hence, if we set

$$k_1^\lambda(z) = \frac{P_Mk_\lambda(z)}{G(\lambda)G(z)},$$

then we get

$$\frac{k_\lambda(z)}{k_1^\lambda(z)} = \frac{1}{1 - l_\lambda(z)} = \sum_{n \geq 0} l_\lambda(z)^n >> 0.$$

Thus we have the contractive inclusion

$$\frac{\mathcal{M}}{G} \subset \mathcal{H}(k).$$
Proof. Recall
\[ M_{\frac{1}{\varphi}} P_0 M_{\frac{1}{\varphi}}^* + \sum M_{u_n} M_{u_n}^* = I. \]

Set
\[ T = P_M - (M_{\frac{1}{\varphi}} P_{\zeta} M M_{\frac{1}{\varphi}}^* + \sum M_{u_n} P_M M_{u_n}^*), \]
then as in the proof of the McCullough-Trent theorem one calculates that \( T \geq 0. \)

Thus
\[ v_\lambda(z) = \langle Tk_\lambda, k_z \rangle \gg 0, \]
calculate some more, the theorem will follow with
\[ l_\lambda(z) = \frac{\varphi(z)\varphi(\lambda)v_\lambda(z)}{G(z)G(\lambda)}. \]
Thm 13 (McCullough-Ri). If $k$ is a Bergman-type kernel, if

$$M \in \text{Lat} (M_\zeta, \mathcal{H}(k)), \ M \neq (0),$$

if

$$\mathcal{C} = M \ominus \zeta M,$$

then

$$M_\zeta | M \text{ is u.e. to } M_\zeta$$
on a space of $\mathcal{C}$-valued analytic functions with operator-valued reproducing kernel of the type

$$K_\lambda(z) = (I_\mathcal{C} - z\bar{\lambda}V(z)V(\lambda)^*) k_\lambda(z),$$

where $V$ is a contractive analytic function

$$V : \mathbb{D} \to \mathcal{B}(\mathcal{E}, \mathcal{C})$$

for some auxiliary Hilbert space $\mathcal{E}$.

This implies that

$$H^2(\mathcal{C}) \subseteq \mathcal{H}(K) \subseteq \mathcal{H}(k, \mathcal{C})$$

contractively.

Here $\mathcal{H}(k, \mathcal{C}) = \mathcal{H}(k) \otimes \mathcal{C}$, i.e. the $\mathcal{C}$-valued $\mathcal{H}(k)$-space.