1. Consider the matrix $A$ given below, and the result $A'$ of applying row reduction to $A$.

$$
A = \begin{bmatrix}
1 & 2 & 0 & -1 \\
3 & 4 & 2 & 2 \\
2 & 2 & 2 & 3
\end{bmatrix} \rightarrow A' = \begin{bmatrix}
1 & 2 & 0 & -1 \\
0 & -2 & 2 & 5 \\
0 & 0 & 0 & 0
\end{bmatrix}.
$$

(i) [10] Find the rank of $A$, and the dimensions of the spaces $\text{Row}(A)$, $\text{N}(A)$, $\text{Col}(A)$, $\text{N}(A^T)$. Justify.

Since $A'$ has two non-zero rows, $\text{rank}(A)=2$, which is the same as the dimensions of $\text{Col}(A)$ and $\text{Row}(A)$; then $\text{N}(A)$ has dimension $4 - 2 = 2$ and $\text{N}(A^T)$ has dimension $3 - 2 = 1$.

(ii) [10] Write down parametric equations for the spaces $\text{Row}(A)$ and $\text{Col}(A)$.

The first two columns (resp. rows) of $A$ (resp. $A'$) give bases for $\text{Col}(A)$ (resp. $\text{Row}(A)$), hence:

$$
\text{Row}(A) = \{s(1, 2, 0, -1) + t(0, -2, 2, 5); s, t \in \mathbb{R}\};
$$

$$
\text{Col}(A) = \{u(1, 3, 2) + v(2, 4, 2); u, v \in \mathbb{R}\}.
$$

(iii) [20] Write down a system of independent defining equations for the space $\text{N}(A)$.

Since $\text{N}(A) = \text{N}(A')$, by definition of nullspace we get from the rows of $A'$ the equations:

$$
x_1 + 2x_2 - x_4 = 0, \quad -2x_2 + 2x_3 + 5x_4 = 0.
$$

2. [20] Let $W$ be the subspace of $\mathbb{R}^5$ defined as the solution space of the homogeneous system:

$$
\begin{cases}
x_1 + x_2 + x_3 + x_4 + 2x_5 = 0 \\
x_1 - x_2 - x_3 = 0
\end{cases}
$$

Find a basis for the orthogonal complement $W^\perp$ of $W$.

The two equations are linearly independent, so $\text{dim } W = 5 - 2 = 3$ and $\text{dim } W^\perp = 5 - 3 = 2$. $W$ is the nullspace of the $2 \times 5$ matrix defining the system, so $W^\perp$ is the row space, with a basis given by the coefficient vectors of the two equations:

$$
\mathcal{B} = \{(1, 1, 1, 1, 2), (2, -1, -1, 0, 0)\}.
$$
3. [20] Find a defining equation for the subspace of $\mathbb{R}^4$ spanned by:

$$\{(1, 0, -1, 1), (0, 1, 1, 2), (0, -1, 1, 0)\}$$

(Hint: These vectors are linearly independent; the general equation for a three-dimensional subspace of $\mathbb{R}^4$ has the form $Ax + By + Cz + Dw = 0$, where you may assume $D = -1$).

As suggested, we write the equation of the subspace as:

$$Ax + By + Cz = w;$$

using the coordinates of the given points yields the system:

$$A - C = 1, \quad B + C = 2, \quad -B + C = 0,$$

which is easily solved for $A, B$ and $C$, giving the equation:

$$2x + y + z - w = 0.$$

4. Let $\mathcal{B}$ be the basis of $\mathbb{R}^2$:

$$\mathcal{B} = \{v_1, v_2\}, \quad v_1 = (1, 2), v_2 = (1, 3).$$

(i)[10] If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by:

$$Tv_1 = (1/2)v_1 + v_2, \quad Tv_2 = 2v_1 - v_2,$$

find $[T]_B$. From the definition of $[T]_B$, we get immediately:

$$[T]_B = \begin{bmatrix} 1/2 & 2 \\ 1 & -1 \end{bmatrix}.$$