LINEAR ALGEBRA - REVIEW PROBLEMS (Eigenvalues, powers of operators)

1. $A$ is $2 \times 2$ symmetric, with eigenvalues $1/2$ and $3$. $E(3)$ is spanned by $(1, 2)$. Let $v = x(1, 2) + y(-2, 1)$. (i) Find $A^n v$ $(n \geq 1$ arbitrary); (ii) What happens to $A^n v$ as $n \to \infty$?

Solution: Since $A$ is symmetric, the two eigenspaces are orthogonal, so $E(1/2)$ is spanned by (say) $(-2, 1)$. Thus $A^n(1, 2) = 3^n(1, 2)$ and $A^n(-2, 1) = (1/2^n)(-2, 1)$. By linearity, $A^n v = 3^n x(1, 2) + (1/2^n)y(-2, 1)$. The component along $(1, 2)$ tends to infinity, while the component along $(-2, 1)$ tends to zero. Thus $A^n v$ approaches the eigenspace $E(3)$ as $n \to \infty$ (and its length tends to infinity).

2. $P : \mathbb{R}^3 \to \mathbb{R}^3$ is the matrix of orthogonal projection onto the plane $x + 2y + z = 0$. (i) Find the eigenvalues and eigenspaces of $P$. (ii) Compute the limit $\lim_{n \to \infty} P^n(1, 1, 1)$.

Solution. Let $E$ be the given plane, $E^\perp$ the orthogonal line. $P v = v$ if $v \in E$, $P v = 0$ if $v \in E^\perp$, and $E$ and $E^\perp$ together span $\mathbb{R}^3$. Hence the eigenvalues are $1$ (with eigenspace $E$) and $0$ (with eigenspace $E^\perp$). (In particular, $P$ is diagonalizable.) The projections of $v = (1, 1, 1)$ on $E^\perp$ and $E$ are (with $u = \frac{1}{\sqrt{6}}(1, 2, 1)$, the unit normal vector to $E$):

$$P^\perp v = \langle v, u \rangle u = \frac{2}{3}(1, 2, 1), \quad P v = v - P^\perp v = (1/3, -1/3, 1/3),$$

and applying $P$ again to $P v$ won’t change it, so $P^n v = (1/3, -1/3, 1/3)$ for all $n \geq 1$.

3. $T : \mathbb{R}^2 \to \mathbb{R}^2$ expands every vector in the plane by a factor of 2, while rotating it by an angle $\pi/4$ (counterclockwise). (i) What are the eigenvalues of $T^2$? (ii) Show that $T^4$ fixes every line through the origin.

Solution.

$$T = 2R_{\pi/4} = \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix},$$

so the eigenvalues are $\sqrt{2} \pm i\sqrt{2}$. For the 4th. power $T^4$, we have $T^4 = 2^4 R_{\pi} = -2^4 I$. Since it is a multiple of the identity, $T^4$ fixes every line through 0.

4. Let $R : \mathbb{R}^3 \to \mathbb{R}^3$ be the rotation matrix with axis spanned by $(1, 1, 2)$, by an angle $\pi/3$ (looking down the axis). (i) What are the eigenvalues of $R^2$? (ii) What is the ‘standard form’ matrix of $R$?
Solution. Vectors $v$ on the axis are fixed by $R$ ($Rv = v$), so 1 is an eigenvector with one-dimensional eigenspace spanned by $(1, 1, 2)$. On the orthogonal plane, $R$ is the rotation matrix:

$$R_{\pi/3} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix},$$

with eigenvalues $(1/2) \pm i\sqrt{3}/2$, which are also (complex) eigenvalues of $R$. The matrix of $R$ in an appropriate basis is given by a rotation block, followed by a 1 on the diagonal:

$$\Lambda = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

5. Let $S : \mathbb{R}^3 \to \mathbb{R}^3$ be reflection on the plane $x + 2y + z = 0$. (i) What are the eigenvalues and eigenspaces of $S$? (ii) Find $S^{2n}(1, 1, 0)$ and $S^{2n+1}(1, 1, 0)$, for each $n \geq 1$.

Solution. $S$ fixes vectors on the given plane (call it $E$), meaning $Sv = v$, and ‘flips’ vectors on the orthogonal line $E^\perp$ (meaning $Sv = -v$). Hence the eigenvalues are 1 (with eigenspace $E$) and $-1$ (with eigenspace $E^\perp$). To find $S(1, 1, 0)$, we decompose $v = (1, 1, 0)$ into components on $E^\perp$ (spanned by the unit vector $u = (1/\sqrt{6}))(1, 2, 1)$ and on $E$:

$$P^\perp = \langle v, u \rangle u = (1/2)(1, 2, 1), \quad Pv = v - P^\perp v = (1/2, 0, -1/2),$$

then compute the action of $S$:

$$Sv = Pv - P^\perp v = (1/2, 0, -1/2) - (1/2, 1, 1/2) = (0, -1, -1).$$

Reflecting twice (or any even number of times) doesn’t move the vector at all (so $S^{2n}v = v$ for all $n \geq 1$), while reflecting an odd number of times is the same as reflecting once, so $S^{2n+1}v = Sv = (0, -1, -1)$ for all $n \geq 1$.

Remark. Here we could have used the standard formula for reflections derived in class, $Sv = v - 2\langle v, u \rangle u = (1, 1, 0) - (1, 2, 1) = (0, -1, -1)$, to find $Sv$. 