MATHEMATICS 251-LINEAR ALGEBRA- FINAL EXAM, 12/15/2005

Instructions. 2h exam- closed book, closed notes. Calculators allowed (but not needed.) No partial credit for answers given without justification, even if correct- show all work! The exam consists of five problems (10pts per item.)

1. The matrix

\[
A = \begin{bmatrix}
0 & 5 & 0 \\
-2 & 6 & 0 \\
0 & 0 & 5
\end{bmatrix}
\]

has 3 + i as an eigenvalue, with eigenvector \( \begin{bmatrix} 2 + i \\ 1 + i \end{bmatrix} \). (In addition, note \( Ae_3 = 5e_3 \).) Find (i) the ‘standard form’ \( \Lambda \) of \( A \), and (ii) an invertible matrix \( P \) so that \( P^{-1}AP = \Lambda \).

2. The characteristic polynomial of the matrix:

\[
B = \begin{bmatrix}
1 & 12 & 0 \\
-1 & 8 & 0 \\
1 & -3 & 4
\end{bmatrix}
\]

is \( p(\lambda) = (\lambda - 4)^2(\lambda - 5) \).

(i) Find bases for the eigenspaces of \( B \). (ii) Is \( B \) diagonalizable? Justify.

3. The 3 \times 3 matrix \( P \) projects vectors in \( \mathbb{R}^3 \) orthogonally onto the plane \( x + 2y + 3z = 0 \).

(i) Find the eigenvalues and eigenspaces of \( P \);
(ii) Find \( P^{1357} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \).

4. A 2 \times 2 symmetric matrix \( C \) has eigenvalues 1/5 and 2. The eigenspace \( E(2) \) is spanned by the vector \( \begin{bmatrix} 2 \\ 3 \end{bmatrix} \). Let \( v = a \begin{bmatrix} 2 \\ 3 \end{bmatrix} + b \begin{bmatrix} -3 \\ 2 \end{bmatrix} \), where \( a, b \) are arbitrary real numbers.

(i) Find \( C^n v \), for arbitrary \( n \geq 0 \);
(ii) Describe the behavior of \( C^n v \) as \( n \to \infty \) (sketch the eigenspaces of \( C \) and the vector \( C^n v \), for large \( n \).)

(Problem 5 is on the back of the page.)
5. (i) Find the maximum and minimum values of the quadratic form in \( \mathbb{R}^2 \):

\[
Q(x, y) = 2x^2 + 4xy - y^2
\]
on the circle \( x^2 + y^2 = 1 \), and the points on the circle where the max/min values are attained.

(ii) Find a rotation of coordinates that eliminates the \( xy \) term in the equation of the conic \( 2x^2 + 4xy - y^2 = -8 \); write the equation of the conic in the new coordinates and identify it (as an ellipse, hyperbola or parabola).

\[(\text{Hint: Recall the new coordinates } x'y' \text{ will be related to the original coordinates } xy \text{ by:})\]

\[
\begin{bmatrix}
x \\
y
\end{bmatrix} = R
\begin{bmatrix}
x' \\
y'
\end{bmatrix},
\]

where \( R \) is a \( 2 \times 2 \) orthogonal matrix that diagonalizes the \( 2 \times 2 \) symmetric matrix associated with \( Q \).\]