Test #3 **Math 231** **Fall 2022** **Name:**

Do all problems and give the **process** of your solution.

1. (30 points). Find the general solution to the following Cauch-Euler equations

(a) $t^2y'' - 3ty' + 3y = 0$

Solution. Solving $r^2 + (-3-1)r + 3 = 0$ or $r^2 - 4r + 3 = 0$ we have $r_1 = 1$ and $r_2 = 3$. So we have two independent solutions $y_1 = t$ and $y_2 = t^3$. So the general solution is

$$y = C_1 t + C_2 t^3$$

(b). $t^2y'' - 3ty' + 3y = t$ (by variation of parameter)

Solution. The general solution $y = \bar{y} + y_p$ where $\bar{y}$ is given in (a). By variation of parameter, $y_p = v_1y_1 + v_2y_2 = v_1t + v_2t^3$ where the derivatives v_1' and v_2' satisfy

$$\begin{cases} v_1'y_1 + v_2'y_2 = 0 \\ \\ v_1'y_1' + v_2'y_2' = \frac{t}{t^2} = t^{-1} \end{cases} \quad \text{or} \quad \begin{cases} v_1't + v_2't^3 = 0 \\ \\ v_1' + 3v_2't^2 = t^{-1} \end{cases}$$

Solving it we have $v_1' = -\frac{1}{2}t^{-1}$ and $v_2' = \frac{1}{2}t^{-3}$. Hence, $v_1 = -\frac{1}{2}\ln t$ and $v_2 = -\frac{1}{4}t^{-2}$. Therefore

$$y_p = -\frac{1}{2}t\ln t - t^3\frac{1}{4}t^{-2} = -\frac{1}{2}t\ln t - \frac{1}{4}t$$

Finally, the general solution is

$$y = C_1t + C_2t^3 - \frac{1}{2}t\ln t - \frac{1}{4}t$$

2. (30 points) Find the Laplace transform $\mathcal{L}(f)$, where

(a).

$$f(t) = \begin{cases} e^t & 0 \le t \le 1 \\ \\ 1 & t > 1 \end{cases}$$

Solution. By definition

$$\mathcal{L}(f) = \int_0^\infty e^{-st}f(t)dt = \int_0^1 e^{-st}e^t dt + \int_1^\infty e^{-st}dt = \int_0^1 e^{-(s-1)t}dt + \frac{1}{s}e^{-s}$$
$$= \frac{1}{s-1}\Big(1 - e^{-(s-1)}\Big) + \frac{1}{s}e^{-s} \qquad (s > 1)$$

(b). $f(t) = t\sin t$

Solution.

$$\mathcal{L}\{t\sin t\} = -\frac{d}{ds}\mathcal{L}\{\sin t\} = -\frac{d}{ds}\frac{1}{s^2+1} = \frac{2s}{(s^2+1)^2}$$

3. (20 points). Find the inverse Laplace transform $\mathcal{L}^{-1}\{\frac{s}{(s^2-1)(s^2+4)}\}$

Solution. By partial fraction decomposition

$$\frac{s}{(s^2-1)(s^2+4)} = \frac{s}{(s-1)(s+1)(s^2+4)} = \frac{1}{10}\frac{1}{s-1} + \frac{1}{10}\frac{1}{s+1} - \frac{1}{5}\frac{s}{s^2+4}$$

Hence,

$$\mathcal{L}^{-1}(\frac{s}{(s^2-1)(s^2+4)}) = \frac{1}{10}\mathcal{L}^{-1}\Big(\frac{1}{s-1}\Big) + \frac{1}{10}\mathcal{L}^{-1}\Big(\frac{1}{s+1}\Big) - \frac{1}{5}\mathcal{L}^{-1}\Big(\frac{s}{s^2+4}\Big)$$
$$= \frac{1}{10}e^t + \frac{1}{10}e^{-t} - \frac{1}{5}\cos(2t)$$

4. (20 points). Solve the following initial value problem $y'' - y = \cos(2t)$, $y(0) = y'(0) = 0$ using the method of Laplace transform.

Solution. Take Laplace transform

$$s^2\mathcal{L}(y) - \mathcal{L}(y) = \mathcal{L}(\cos(2t)) = \frac{s}{s^2+4}$$

$$\mathcal{L}(y) = \frac{s}{(s^2-1)(s^2+4)}$$

Hence, by the result from Problem 3

$$y = \mathcal{L}^{-1}\Big(\frac{s}{(s^2-1)(s^2+4)}\Big) = \frac{1}{10}e^t + \frac{1}{10}e^{-t} - \frac{1}{5}\cos(2t)$$