Test #2 **Math 231 (10:30-11:20)** **Fall 2021** **Name:**

Solve the given equations in the problem 1–4.

1. (20 point). $y'' - 6y' + 9y = 0$, $y(0) = 1$, $y'(0) = 5$.

Solution. $r^2 - 6r + 9 = 0$ has a double root $r = 3$. General solution is $y = (C_1 + C_2t)e^{3t}$, $y' = \big((3C_1 + C_2) + 3C_2t\big)e^{3t}$.

$t = 0$: $C_1 = 1$, $C_2 = 2$. So $y = (1 + 2t)e^{3t}$.

2. (20 point).$y'' - y = 0$, $\quad y(0) = 0$, $y'(0) = 5$.

Solution. $r^2 - 1 = 0$ has roots $r = \pm 1$. General solution: $y = C_1e^{-t} + C_2e^t$. $y_1' = -C_1e^{-t} + C_2e^t$

$t = 0$: $C_1 + C_2 = 0$, $-C_1 + C_2 = 5$. We have $C_1 = -5/2$, $C_2 = 5/2$. The special solution is

$$y = -\frac{5}{2}e^{-t} + \frac{5}{2}e^t$$

3. (20 point). $y'' - 2y' + 2y = 0$.

Solution. $r^2 - 2r + 2 = 0$ has complex roots $r = 1 \pm i$. The general solution is $y = C_1e^t \cos t + C_2e^t \sin t$.

4. (20 point). $y'' - 2y' + 2y = \sin t$.

Solution. The general solution is $y = \bar{y} + y_p$. From Problem 4, $\bar{y} = C_1e^t \cos t + C_2e^t \sin t$. Set $y_p = A\cos t + B\sin t$. Then $y_p' = -A\sin t + B\cos t$ and $y_p'' = -A\cos t - B\sin t$. Hence

$$\begin{aligned} &y_p'' - 2y_p' + 2y_p = (-A\cos t - B\sin t) - 2(-A\sin t + B\cos t) + 2(A\cos t + B\sin t) \\ &= (-A - 2B + 2A)\cos t + (-B + 2A + 2B)\sin t = (A - 2B)\cos t + (2A + B)\sin t \end{aligned}$$

So we have $A - 2B = 0$ and $2A + B = 1$. Or, $A = 2/5$ and $B = 1/5$. Hence

$$y_p = \frac{2}{5}\cos t + \frac{1}{5}\sin t$$

The general solution is

$$y = C_1e^t \cos t + C_2e^t \sin t + \frac{2}{5}\cos t + \frac{1}{5}\sin t$$

5. (20 point). A brine solution of salt flows at a constant rate of 8L/min into a tank that initially held 100 L of brine solution in which was dissolved 0.5kg of salt. The solution inside the tank is kept well stirred and flows out at the same rate. If the concentration of the salt in the brine entering the tank is 0.05kg/L, determine the mass of the salt in the tank after t min.

Solution. Let $x(t)$ be the mass of the salt after t min. We have

$$\begin{cases} \frac{dx}{dt} = 8 \times 0.05 - 8 \times \frac{x}{100} = 0.4 - 0.08x \\ x(0) = 0.5 \end{cases}$$

Hence

$$\frac{dx}{0.4 - 0.08x} = dt, \qquad \frac{-1}{0.08} \ln(0.4 - 0.08x) = t + C, \qquad 0.4 - 0.08x = Ce^{-0.08t}$$

Let $t = 0$ gets $C = 0.36$. Thus, $x(t) = 5 - 4.5e^{-0.08t}$