MODERATE AND SMALL DEVIATIONS FOR THE RANGES OF ONE-DIMENSIONAL RANDOM WALKS

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We establish the moderate and small deviations for the ranges of the integer valued random walks. Our theorems apply to the limsup and the liminf laws of the iterated logarithm.

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1. Introduction.

Let $\{S(n)\}_{n\geq 0}$ be an integer valued random walk. That is,

$$S(0) = 0 \quad \text{and} \quad S(n) = X(1) + \cdots + X(n) \qquad n = 1, 2, \cdots$$

where $\{X;\ X(k);\ k \geq 1\}$ is an i.i.d. integer valued sequence. Throughout we assume that

$$\mathbb{E}\, X = 0 \quad \text{and} \quad \sigma^2 \equiv \mathbb{E}\, X^2 < \infty. \tag{1.1}$$

We also assume that the smallest group that supports $\{S(n)\}_{n\geq 0}$ is $\mathbb{Z}$.

For any $\Delta \subset \mathbb{R}^+$, we set

$$S(\Delta) = \{S(k); \qquad k \in \Delta\}.$$

Let $\{S_1(n)\}_{n\geq 0}, \cdots, \{S_p(n)\}_{n\geq 0}$ be independent copies of $\{S(n)\}_{n\geq 0}$. In this paper we study the random sequences

$$\#\big\{S[0,n]\big\} \quad \text{and} \quad \#\big\{S_1[0,n] \cap \cdots \cap S_p[0,n]\big\} \qquad n = 1, 2, \cdots.$$

This work appears as a part of recent effort made by the author and his coauthors ([1]-[3], [5]-[9]) in the study of exponential asymptotics arising from sample path intersections.

To simplify our notation we allow $p = 1$, in which case the range intersection $S_1[0,n] \cap \cdots \cap S_p[0,n]$ is identified with the range $S[0,n]$.

Throughout, $\{b_n\}$ is a positive sequence satisfying

$$b_n \longrightarrow \infty \quad \text{and} \quad b_n = o(n) \qquad (n \to \infty). \tag{1.2}$$

Theorem 1. *For any $\lambda > 0$ and integer $p \geq 1$,*

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{\#\big\{S_1[0,n] \cap \cdots \cap S_p[0,n]\big\} \geq \lambda\sqrt{nb_n}\Big\} = -\frac{p\lambda^2}{2\sigma^2}. \tag{1.3}$$

Taking $p = 1$ in Theorem 1 we have

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{\#\big\{S[0,n]\big\} \geq \lambda\sqrt{nb_n}\Big\} = -\frac{\lambda^2}{2\sigma^2}. \tag{1.4}$$

We point out the classic fact that the random walk $S(n)$ satisfies the same moderate deviation under stronger moment condition. A striking fact about Theorem 1 is that the moderate deviation holds for the range and the intersection of the ranges under no moment assumption rather than (1.1).

Theorem 2. *For any $\lambda > 0$,*

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{\#\big\{S[0,n]\big\} \leq \lambda\sqrt{\frac{n}{b_n}}\Big\} = -\frac{\pi^2\sigma^2}{2\lambda^2}. \tag{1.5}$$

The small deviation of this type does not hold for intersection of the ranges. Indeed, the recent paper by Kleke and Mörters (2004) on the lower tail of the intersections of the Brownian paths suggests that the probability

$$\mathbb{P}\Big\{\#\big\{S_1[0,n] \cap \cdots \cap S_p[0,n]\big\} \leq \lambda\sqrt{\frac{n}{b_n}}\Big\}$$

does not have exponential decay as $p > 1$.

Theorem 1 and Theorem 2 apply, respectively, to the limsup and the liminf laws of the iterated logarithm given in the following theorem.

Theorem 3. *For any integer $p \geq 1$,*

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n\log\log n}} \#\big\{S_1[0,n]\cap\cdots\cap S_p[0,n]\big\} = \sqrt{\frac{2}{p}}\sigma \qquad a.s. \tag{1.6}$$

In the case $p = 1$,

$$\liminf_{n\to\infty} \sqrt{\frac{\log\log n}{n}} \#\big\{S[0,n]\big\} = \frac{\pi}{\sqrt{2}}\sigma \qquad a.s. \tag{1.7}$$

The study shows that the asymptotic behaviors of the range and of the intersection of the ranges have strong dimension dependence. It was proved in Dvoretzky and Erdös (1951) that the range $\#\big\{S[0,n]\big\}$ of a d-dimension centered, square integrable lattice-valued random walk $\{S(n)\}_{n\geq 0}$ satisfies the strong law of large numbers as $d \geq 2$, contrary to what is given in Theorem 3. The central limit theorem shows that $\#\big\{S[0,n]\big\} - \mathbb{E}\,\#\big\{S[0,n]\big\}$ is asymptotically normal as $d \geq 3$ (Jain and Pruitt (1971, 1974)), and is attracted by the renormalized self-intersection local time of a planar Brownian motion as $d = 2$ (Le Gall (1986a)). See Le Gall and Rosen (1991) for a central limit theorem when the random walk is in the domain of attraction of a stable law. The LIL for $\#\big\{S[0,n]\big\} - \mathbb{E}\,\#\big\{S[0,n]\big\}$ can be found in Jain and Pruitt (1972) and Hamana (1998) as $d \geq 4$; and in Bass and Kumagai (2002) as $d = 2, 3$.

As for the intersection of the independent ranges, it is well known (Dvoretzky and Erdös (1951), (1954)) that p independent paths of the centered and squared integrable $\mathbb{Z}^d$-valued random walks $\{S_1(n)\}_{n\geq 0}, \cdots, \{S_p(n)\}_{n\geq 0}$ intersect infinitely often if and only if $p(d-2) \leq d$. Under this condition, the laws of weak convergence were established in Le Gall (1986a, 1986b). The moderate deviations for $\#\big\{S_1[0,n]\cap\cdots\cap S_p[0,n]\big\}$, or the related laws of the iterated logarithm, can be found in Marcus and Rosen (1997) for $d = 4$ and $p = 2$; in Rosen (1997) for $d = p = 3$; and in the author's recent paper (Chen (2004)) for $d = 2$ and $p \geq 2$ and for $d = 3$ and $p = 2$.

The case $d = 1$ is radically different from the multi-dimensional case where the range and the intersection of the independent ranges have different orders of upper tail asymptotics. While the asymptotic behaviors of the range are closely related to that of the intersection local time as $d \geq 2$, the 1-dimensional case is largely determined by the relation

$$S[0,n] \subset \big[\min_{0\leq k\leq n} S(k),\ \max_{0\leq k\leq n} S(k)\big] \tag{1.8}$$

where, as a convention in this paper, the right hand side represents a interval containing only integers. Indeed, the equality holds in the case of the simple random walks. In the general case, it is known (Le Gall (1986a)) that the possible "holes" within the interval on the right hand side of (1.8) are insignificant in the sense of weak convergence. The case of the moderate deviations appears to be more delicate. Indeed, the moderate deviation for the connected range $\max_{0\leq k\leq n} S(k) - \min_{0\leq k\leq n} S(k)$ can fail without assuming an extra moment condition as strong as exponential integrability. This fact suggests that in general, the range is not exponentially close to the connected range.

2. Upper bound in Theorem 1.

In this section we prove that

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\Big\{\#\big\{S_1[0,n]\cap\cdots\cap S_p[0,n]\big\}\ge\lambda\sqrt{nb_n}\Big\}\le-\frac{p\lambda^2}{2\sigma^2}. \tag{2.1}$$

By Chebyshev inequality, we only need to show that

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\,\exp\Big\{\theta\sqrt{\frac{b_n}{n}}\#\big\{S_1[0,n]\cap\cdots\cap S_p[0,n]\big\}\Big\}\le\frac{\sigma^2\theta^2}{2p}. \tag{2.2}$$

To this end we first prove

Lemma 2.1. *For any $\theta>0$,*

$$\sup_n\mathbb{E}\,\exp\Big\{\frac{\theta}{\sqrt{n}}\#\big\{S[0,n]\big\}\Big\}<\infty. \tag{2.3}$$

Proof. Recall ((4.8), Chen (2004)) that for any $a,b>0$,

$$\mathbb{P}\Big\{\#\big\{S[0,n]\big\}\ge a+b\Big\}\le\mathbb{P}\Big\{\#\big\{S[0,n]\big\}\ge a\Big\}\mathbb{P}\Big\{\#\big\{S[0,n]\big\}\ge b\Big\}.$$

We therefore have that for any $C>0$ and the integer $m\ge1$,

$$\mathbb{P}\Big\{\#\big\{S[0,n]\big\}\ge Cm\sqrt{n}\Big\}\le\bigg(\mathbb{P}\Big\{\#\big\{S[0,n]\big\}\ge C\sqrt{n}\Big\}\bigg)^m.$$

By the fact that $\mathbb{E}\,\#\big\{S[0,n]\big\}=O\big(\sqrt{n}\big)$ one can take $C>0$ large enough so that

$$\sup_n\mathbb{P}\Big\{\#\big\{S[0,n]\big\}\ge C\sqrt{n}\Big\}\le e^{-2}.$$

Therefore, (2.3) holds for $\theta=C^{-1}$. We now show that it holds for all $\theta>0$. Indeed, take $\delta>0$ such that $\delta<(C\theta)^{-2}$ and write $k_n=[\delta n]$. The desired conclusion follows from the following estimate:

$$\begin{aligned}&\mathbb{E}\,\exp\Big\{\frac{\theta}{\sqrt{n}}\#\big\{S[0,n]\big\}\Big\}\le\bigg(\mathbb{E}\,\exp\Big\{\frac{\theta}{\sqrt{n}}\#\big\{S[0,k_n]\big\}\Big\}\bigg)^{[\delta^{-1}]+1}\\&\le\bigg(\mathbb{E}\,\exp\Big\{\frac{C^{-1}}{\sqrt{k_n}}\#\big\{S[0,k_n]\big\}\Big\}\bigg)^{[\delta^{-1}]+1}.\end{aligned}$$

□

We first establish (2.2) in the case $p=1$. Let $t>0$ be arbitrary but fixed and write $t_n=[tn/b_n]$ and $\gamma_n=[n/t_n]$. Notice that

$$\#\big\{S[0,n]\big\}\le\sum_{i=1}^{\gamma_n+1}\#\big\{S[(i-1)t_n,it_n]\big\}$$

and that the sequence

$$\#\{S[(i-1)t_n, it_n]\} \qquad i = 1, \cdots, \gamma_n + 1$$

is an i.i.d. So for any $\theta > 0$,

$$\mathbb{E}\exp\Big\{\theta\sqrt{\frac{b_n}{n}}\#\{S[0,n]\}\Big\} \le \Big(\mathbb{E}\exp\Big\{\theta\sqrt{\frac{b_n}{n}}\#\{S[0,t_n]\}\Big\}\Big)^{\gamma_n+1}.$$

According to Theorem 4.1 in Le Gall (1986a),

$$\sqrt{\frac{b_n}{n}}\#\{S[0,t_n]\} \xrightarrow{d} \sigma\sqrt{t}\Big(\max_{0\le s\le 1} W(s) - \min_{0\le s\le 1} W(s)\Big) \tag{2.4}$$

as $n \to \infty$, where $W(s)$ is an one dimensional Brownian motion. By Lemma 2.1 and the dominated convergence theorem,

$$\mathbb{E}\exp\Big\{\theta\sqrt{\frac{b_n}{n}}\#\{S[0,t_n]\}\Big\} \longrightarrow \mathbb{E}\exp\Big\{\sigma\theta\sqrt{t}\big(\max_{0\le s\le 1} W(s) - \min_{0\le s\le 1} W(s)\big)\Big\} \qquad (n\to\infty).$$

Hence,

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\Big\{\theta\sqrt{\frac{b_n}{n}}\#\{S[0,n]\}\Big\}\\ &\le \frac{1}{t}\log\mathbb{E}\exp\Big\{\sigma\theta\sqrt{t}\big(\max_{0\le s\le 1} W(s) - \min_{0\le s\le 1} W(s)\big)\Big\}.\end{aligned} \tag{2.5}$$

Let

$$H = \Big\{x \in C_0[0,1];\ x(s) \text{ is absolutely continuous on } [0,1] \text{ and } \int_0^1 |\dot{x}(s)|^2 ds < \infty\Big\}.$$

By the large deviation principle for Brownian motions (Theorem 5.2.3 in Dembo and Zeitouni (1998)),

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\Big\{\sigma\theta\sqrt{t}\big(\max_{0\le s\le 1} W(s) - \min_{0\le s\le 1} W(s)\big)\Big\}\\ &= \sup_{x\in H}\Big\{\sigma\theta\Big(\max_{0\le s\le 1} x(s) - \min_{0\le s\le 1} x(s)\Big) - \frac{1}{2}\int_0^1 |\dot{x}(s)|^2 ds\Big\}.\end{aligned} \tag{2.6}$$

Applying the mean value theorem and Cauchy's inequality gives

$$\max_{0\le s\le 1} x(s) - \min_{0\le s\le 1} x(s) \le \Big(\int_0^1 |\dot{x}(s)|^2 ds\Big)^{1/2}.$$

Therefore, the right hand side of (2.6) is bounded by

$$\sup_{\lambda>0}\big\{\sigma\theta\lambda - \frac{1}{2}\lambda^2\big\} = \frac{1}{2}\sigma^2\theta^2.$$

Letting $t \to \infty$ in (2.5) we have

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\Big\{\theta\sqrt{\frac{b_n}{n}}\#\{S[0,n]\}\Big\} \le \frac{1}{2}\sigma^2\theta^2. \tag{2.7}$$

We now prove (2.2) in the case $p > 1$. By the fact

$$\#\{S_1[0,n]\cap\cdots\cap S_p[0,n]\} = \sum_{x\in\mathbb{Z}}\prod_{j=1}^p 1_{\{x\in S_j[0,n]\}} \le \sum_{x\in\mathbb{Z}}\frac{1}{p}\sum_{j=1}^p 1_{\{x\in S_j[0,n]\}} = \frac{1}{p}\sum_{j=1}^p \#\{S_j[0,n]\}$$

and by independence,

$$\mathbb{E}\exp\Big\{\theta\sqrt{\frac{b_n}{n}}\#\{S_1[0,n]\cap\cdots\cap S_p[0,n]\}\Big\} \le \Big(\mathbb{E}\exp\Big\{\frac{\theta}{p}\sqrt{\frac{b_n}{n}}\#\{S[0,n]\}\Big\}\Big)^p.$$

By (2.7) (with θ being replaced by θp^{-1}) we have (2.2).

3. Lower bound in Theorem 1.

In this section we prove that

$$\liminf_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\Big\{\#\{S_1[0,n]\cap\cdots\cap S_p[0,n]\}\ge\lambda\sqrt{nb_n}\Big\}\ge -\frac{p\lambda^2}{2\sigma^2}. \tag{3.1}$$

Let $M > 0$ be a deterministic number and write

$$\bar{S}(0) = 0 \quad\text{and}\quad \bar{S}(n) = X(1)1_{\{|X(1)|\le M\}} + \cdots + X(n)1_{\{|X(n)|\le M\}} \qquad n = 1,2,\cdots.$$

We may define the independent copies $\{X_1(k)\}_{k\ge 1},\cdots,\{X_p(k)\}_{k\ge 1}$ of $\{X(k)\}_{k\ge 1}$ and let

$$\bar{S}_j(n) = X_j(1) + \cdots + X_j(n) \qquad n = 1,2,\cdots$$

$$\bar{S}_j(0) = 0 \quad\text{and}\quad \bar{S}_j(n) = X_j(1)1_{\{|X_j(1)|\le M\}} + \cdots + X_j(n)1_{\{|X_j(n)|\le M\}} \qquad n = 1,2,\cdots$$

for each $j = 1,\cdots,p$.

Lemma 3.1. *For any $t > 0$ and $n \ge 1$,*

$$\begin{aligned}&\mathbb{P}\{\#\{S_1[0,n]\cap\cdots\cap S_p[0,n]\}\ge t\}\\ &\ge \Big(1+\mathbb{P}\{|X|>M\}\Big)^{-np}\mathbb{P}\{\#\{\bar{S}_1[0,n]\cap\cdots\cap\bar{S}_p[0,n]\}\ge t\}.\end{aligned}$$

Proof. Write $[1,n] = \{1,\cdots n\}$. For any $A\subset[1,n]$, let $\{k_1,\cdots,k_{n-|A|}\} = [1,n]\setminus A$ and for any $1\le j\le p$, write

$$S_j^A(0) = 0 \quad\text{and}\quad S_j^A(i) = X_j(k_1)+\cdots X_j(k_i) \qquad i = 1,\cdots,n-|A|$$

where $|A| = \#(A)$. We have

$$\begin{aligned}&\mathbb{P}\{\#\{\bar{S}_1[0,n]\cap\cdots\cap\bar{S}_p[0,n]\}\ge t\}\\
&= \sum_{A_1,\cdots,A_p\subset[1,n]}\mathbb{P}\{|X_1(l_1)|>M,\cdots,|X_p(l_p)|>M\ \forall(l_1,\cdots,l_p)\in A_1\times\cdots\times A_p;\\
&\qquad |X_1(m_1)|\le M,\cdots,|X_p(m_p)|\le M\ \forall(m_1,\cdots,m_p)\in([1,n]\setminus A_1)\times\cdots\times([1,n]\setminus A_1);\\
&\qquad \#\{S_1^{A_1}[0,n-|A_1|]\cap\cdots\cap S_p^{A_p}[0,n-|A_p|]\}\ge t\}\\
&\le \sum_{A_1,\cdots,A_p\subset[1,n]}\mathbb{P}\{|X_1(l_1)|>M,\cdots,|X_p(l_p)|>M\ \forall(l_1,\cdots,l_p)\in A_1\times\cdots\times A_p;\\
&\qquad \#\{S_1^{A_1}[0,n-|A_1|]\cap\cdots\cap S_p^{A_p}[0,n-|A_p|]\}\ge t\}\\
&= \sum_{A_1,\cdots,A_p\subset[1,n]}\Big(\mathbb{P}\{|X|>M\}\Big)^{|A_1|+\cdots+|A_p|}\mathbb{P}\{\#\{S_1[0,n-|A_1|]\cap\cdots\cap S_p[0,n-|A_p|]\}\ge t\}\\
&\le \Big(1+\mathbb{P}\{|X|>M\}\Big)^{np}\mathbb{P}\{\#\{S_1[0,n]\cap\cdots\cap S_p[0,n]\}\ge t\}.\end{aligned}$$

$\square$

By the assumption (1.1), we can find a positive deterministic sequence $\{\alpha_n\}$ such that

$$\alpha_n \longrightarrow 0, \quad \frac{n}{b_n}\mathbb{P}\Big\{|X| > \alpha_n\sqrt{\frac{n}{b_n}}\Big\} \longrightarrow 0, \quad \sqrt{\frac{n}{b_n}}\mathbb{E}\,|X|1_{\{|X|>\alpha_n\sqrt{nb_n^{-1}}\}} \longrightarrow 0 \tag{3.2}$$

as $n \to \infty$. In the rest of this section, we take $M = M_n = \alpha_n\sqrt{nb_n^{-1}}$ and let the random walks $\bar{S}(k); \bar{S}_1(k), \cdots, \bar{S}_p(k)$ be defined as before. By Lemma 3.1 and (3.2),

$$\begin{aligned}
&\mathbb{P}\Big\{\#\{S_1[0,n] \cap \cdots \cap S_p[0,n]\} \geq \lambda\sqrt{nb_n}\Big\} \geq \\
&\geq \Big\{1 + o\Big(\frac{b_n}{n}\Big)\Big\}^{-np}\mathbb{P}\Big\{\#\{\bar{S}_1[0,n] \cap \cdots \cap \bar{S}_p[0,n]\} \geq \lambda\sqrt{nb_n}\Big\}.
\end{aligned}$$

To prove (3.1), therefore, we need only to establish that

$$\liminf_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\Big\{\#\{\bar{S}_1[0,n] \cap \cdots \cap \bar{S}_p[0,n]\} \geq \lambda\sqrt{nb_n}\Big\} \geq -\frac{p\lambda^2}{2\sigma^2}.$$

Write $t_n = [n/b_n]$ and $\gamma_n = [n/t_n]$. By the fact that $\gamma_n t_n \leq n$, it suffices to show

$$\liminf_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\Big\{\#\{\bar{S}_1[0,\gamma_n t_n] \cap \cdots \cap \bar{S}_p[0,\gamma_n t_n]\} \geq \lambda\sqrt{nb_n}\Big\} \geq -\frac{p\lambda^2}{2\sigma^2}. \tag{3.3}$$

Notice that

$$\begin{aligned}
&\Big[\max_{1\leq j\leq p}\min_{0\leq k\leq\gamma_n t_n}\bar{S}_j(k),\ \min_{1\leq j\leq p}\max_{0\leq k\leq\gamma_n t_n}\bar{S}_j(k)\Big] \setminus \Big(\bar{S}_1[0,\gamma_n t_n] \cap \cdots \cap \bar{S}_p[0,\gamma_n t_n]\Big) \\
&\subset \bigcup_{j=1}^{p}\Big(\big[\min_{0\leq k\leq\gamma_n t_n}\bar{S}_j(k),\ \max_{0\leq k\leq\gamma_n t_n}\bar{S}_j(k)\big] \setminus \bar{S}_j[0,n]\Big).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\Big(\min_{1\leq j\leq p}\max_{0\leq k\leq\gamma_n t_n}\bar{S}_j(k) - \max_{1\leq j\leq p}\min_{0\leq k\leq\gamma_n t_n}\bar{S}_j(k)\Big) - \#\{\bar{S}_1[0,\gamma_n t_n] \cap \cdots \cap \bar{S}_p[0,\gamma_n t_n]\} \\
&\leq \sum_{j=1}^{p}\Big(\big\{\max_{0\leq k\leq\gamma_n t_n}\bar{S}_j(k) - \min_{0\leq k\leq\gamma_n t_n}\bar{S}_j(k)\big\} - \#\{\bar{S}_j[0,\gamma_n t_n]\}\Big).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathbb{P}\{\#\{\bar{S}_1[0,\gamma_n t_n|] \cap \cdots \cap \bar{S}_p[0,\gamma_n t_n]\} \geq \lambda\sqrt{nb_n}\} \\
&+ p\mathbb{P}\{\{\max_{0\leq k\leq\gamma_n t_n}\bar{S}(k) - \min_{0\leq k\leq\gamma_n t_n}\bar{S}(k)\} - \#\{\bar{S}[0,\gamma_n t_n]\} \geq p^{-1}\epsilon\sqrt{nb_n}\} \\
&\geq \mathbb{P}\Big\{\min_{1\leq j\leq p}\max_{0\leq k\leq\gamma_n t_n}\bar{S}_j(k) - \max_{1\leq j\leq p}\min_{0\leq k\leq\gamma_n t_n}\bar{S}_j(k) \geq (\lambda+\epsilon)\sqrt{nb_n}\Big\}.
\end{aligned}$$

By the fact that

$$\min_{1\leq j\leq p}\max_{0\leq k\leq\gamma_n t_n}\bar{S}_j(k) - \max_{1\leq j\leq p}\min_{0\leq k\leq\gamma_n t_n}\bar{S}_j(k) \geq \min_{1\leq j\leq p}\max_{0\leq k\leq\gamma_n t_n}\bar{S}_j(k) \geq \min_{1\leq j\leq p}\bar{S}_j(\gamma_n t_n)$$

we have

$$\begin{aligned}
&\mathbb{P}\big\{\#\big\{\bar S_1[0,\gamma_n t_n|]\cap\cdots\cap \bar S_p[0,\gamma_n t_n]\big\}\ge \lambda\sqrt{nb_n}\big\}\\
&+p\mathbb{P}\big\{\big\{\max_{0\le k\le\gamma_n t_n}\bar S(k)-\min_{0\le k\le\gamma_n t_n}\bar S(k)\big\}-\#\big\{\bar S[0,\gamma_n t_n]\big\}\ge p^{-1}\epsilon\sqrt{nb_n}\big\}\\
&\ge\bigg(\mathbb{P}\Big\{\bar S(\gamma_n t_n)\ge(\lambda+\epsilon)\sqrt{nb_n}\Big\}\bigg)^p.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\max\bigg\{\liminf_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\Big\{\#\big\{\bar S_1[0,\gamma_n t_n]\cap\cdots\cap\bar S_p[0,\gamma_n t_n]\big\}\ge\lambda\sqrt{nb_n}\Big\},\\
&\qquad\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\big\{\big\{\max_{0\le k\le\gamma_n t_n}\bar S(k)-\min_{0\le k\le\gamma_n t_n}\bar S(k)\big\}-\#\big\{\bar S[0,\gamma_n t_n]\big\}\ge p^{-1}\epsilon\sqrt{nb_n}\big\}\bigg\}\\
&\ge p\liminf_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\Big\{\bar S(\gamma_n t_n)\ge(\lambda+\epsilon)\sqrt{nb_n}\Big\}.
\end{aligned}\tag{3.4}$$

Notice that for any real number θ,

$$\begin{aligned}
&\mathbb{E}\,\exp\Big\{\theta\sqrt{\frac{b_n}{n}}\big(\bar S(\gamma_n t_n)-\mathbb{E}\,\bar S(\gamma_n t_n)\big)\Big\}\\
&=\bigg(\mathbb{E}\,\exp\Big\{\theta\sqrt{\frac{b_n}{n}}\big(X1_{\{|X|\le M_n\}}-\mathbb{E}\,X1_{\{|X|\le M_n\}}\big)\Big\}\bigg)^{\gamma_n t_n}\\
&=\Big\{1+\frac{b_n}{2n}\sigma^2\theta+o\Big(\frac{b_n}{n}\Big)\Big\}^{\gamma_n t_n}\qquad(n\to\infty)
\end{aligned}$$

where the last step follows from (3.2) and Taylor's expansion. Therefore,

$$\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\,\exp\Big\{\theta\sqrt{\frac{b_n}{n}}\big(\bar S(\gamma_n t_n)-\mathbb{E}\,\bar S(\gamma_n t_n)\big)\Big\}=\frac12\sigma^2\theta^2.$$

By Gärtner-Ellis theorem (Theorem 2.3.6 in Dembo and Zeitouni (1998)),

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\Big\{\bar S(\gamma_n t_n)-\mathbb{E}\,\bar S(\gamma_n t_n)\ge\lambda\sqrt{nb_n}\Big\}\\
&=-\sup_\theta\Big\{\lambda\theta-\frac12\sigma^2\theta^2\Big\}=-\frac{\lambda^2}{2\sigma^2}\qquad(\lambda>0).
\end{aligned}\tag{3.5}$$

On the other hand, from (3.2) we have

$$\begin{aligned}
&\big|\mathbb{E}\,\bar S(\gamma_n t_n)\big|=n\big|\mathbb{E}\,X1_{\{|X|\le\alpha_n\sqrt{nb_n^{-1}}\}}\big|\\
&=n\big|\mathbb{E}\,X1_{\{|X|>\alpha_n\sqrt{nb_n^{-1}}\}}\big|\le n\mathbb{E}\,|X|1_{\{|X|>\alpha_n\sqrt{nb_n^{-1}}\}}=o\big(\sqrt{nb_n}\big)\qquad(n\to\infty).
\end{aligned}$$

Hence, (3.5) remains true as $\bar S(\gamma_n t_n)-\mathbb{E}\,\bar S(\gamma_n t_n)$ is replaced by $\bar S(\gamma_n t_n)$. In particular,

$$\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\Big\{\bar S(\gamma_n t_n)\ge(\lambda+\epsilon)\sqrt{nb_n}\Big\}=-\frac{(\lambda+\epsilon)^2}{2\sigma^2}.\tag{3.6}$$

In view of (3.4) and (3.6), we will have (3.3) if

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\Big\{\big\{\max_{0\le k\le\gamma_n t_n}\bar{S}(k)-\min_{0\le k\le\gamma_n t_n}\bar{S}(k)\big\}-\#\big\{\bar{S}[0,\gamma_n t_n]\big\}\ge\epsilon\sqrt{nb_n}\Big\}=-\infty \tag{3.7}$$

holds for any $\epsilon>0$.

Indeed,

$$\begin{aligned}
&\big[\min_{0\le k\le\gamma_n t_n}\bar{S}(k),\ \max_{0\le k\le\gamma_n t_n}\bar{S}(k)\big]\setminus\bar{S}[0,\gamma_n t_n]\\
&\subset\bigcup_{i=1}^{[b_n]}\Big(\big[\min_{(i-1)\gamma_n\le k\le i\gamma_n}\bar{S}(k),\ \max_{(i-1)\gamma_n\le k\le i\gamma_n}\bar{S}(k)\big]\setminus\bar{S}[(i-1)\gamma_n,\ i\gamma_n]\Big).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\big\{\max_{0\le k\le\gamma_n t_n}\bar{S}(k)-\min_{0\le k\le\gamma_n t_n}\bar{S}(k)\big\}-\#\big\{\bar{S}[0,\gamma_n t_n]\big\}\\
&\le\sum_{i=1}^{\gamma_n}\Big(\big\{\max_{(i-1)t_n\le k\le it_n}\bar{S}(k)-\min_{(i-1)t_n\le k\le it_n}\bar{S}(k)\big\}-\#\big\{\bar{S}[(i-1)t_n,\ it_n]\big\}\Big)\\
&=\sum_{i=1}^{\gamma_n}\xi_i\qquad\text{(say)}.
\end{aligned}$$

Notice that $\xi_1,\cdots\xi_{\gamma_n}$ are i.i.d. non-negative random variables. For any $\theta>0$,

$$\begin{aligned}
&\mathbb{E}\exp\Big\{\theta\sqrt{\frac{b_n}{n}}\big\{\max_{0\le k\le\gamma_n t_n}\bar{S}(k)-\min_{0\le k\le\gamma_n t_n}\bar{S}(k)\big\}-\#\big\{\bar{S}[0,\gamma_n t_n]\big\}\Big\}\\
&\le\bigg(\mathbb{E}\exp\Big\{\theta\sqrt{\frac{b_n}{n}}\Big(\big\{\max_{0\le k\le t_n}\bar{S}(k)-\min_{0\le k\le t_n}\bar{S}(k)\big\}-\#\big\{\bar{S}[0,\ t_n]\big\}\Big)\Big\}\bigg)^{\gamma_n}.
\end{aligned}\tag{3.8}$$

By the fact that

$$\#\big\{S[0,\ t_n]\big\}\le\max_{0\le k\le t_n}S(k)-\min_{0\le k\le t_n}S(k),$$

by the invariant principle

$$\sqrt{\frac{b_n}{n}}\mathbb{E}\Big(\max_{0\le k\le t_n}S(k)-\min_{0\le k\le t_n}S(k)\Big)\longrightarrow\mathbb{E}\Big(\max_{0\le s\le 1}W(s)-\min_{0\le s\le 1}W(s)\Big)\qquad(n\to\infty)$$

and by (4.g) in Le Gall (1986a) which leads to

$$\sqrt{\frac{b_n}{n}}\mathbb{E}\,\#\big\{S[0,\ t_n]\big\}\longrightarrow\mathbb{E}\Big(\max_{0\le s\le 1}W(s)-\min_{0\le s\le 1}W(s)\Big)\qquad(n\to\infty),$$

we must have

$$\sqrt{\frac{b_n}{n}}\Big(\big\{\max_{0\le k\le t_n}S(k)-\min_{0\le k\le t_n}S(k)\big\}-\#\big\{S[0,\ t_n]\big\}\Big)\xrightarrow{P}0.$$

Notice that for any $\epsilon > 0$,

$$\mathbb{P}\Big\{\big\{\max_{0\le k\le t_n}\bar S(k) - \min_{0\le k\le t_n}\bar S(k)\big\} - \#\big\{\bar S[0,\ t_n]\big\} \ge \epsilon\sqrt{\frac{n}{b_n}}\Big\}$$
$$\le \mathbb{P}\Big\{\big\{\max_{0\le k\le t_n} S(k) - \min_{0\le k\le t_n} S(k)\big\} - \#\big\{S[0,\ t_n]\big\} \ge \epsilon\sqrt{\frac{n}{b_n}}\Big\} + \frac{n}{b_n}\mathbb{P}\Big\{|X| > \alpha_n\sqrt{\frac{n}{b_n}}\Big\}.$$

In view of (3.2) we obtain

$$\sqrt{\frac{b_n}{n}}\Big(\big\{\max_{0\le k\le t_n}\bar S(k) - \min_{0\le k\le t_n}\bar S(k)\big\} - \#\big\{\bar S[0,\ t_n]\big\}\Big) \xrightarrow{P} 0. \tag{3.9}$$

By the facts that $|X| \le C\sqrt{n/b_n}$ and that $\sqrt{b_n/n}\bar S(t_n)$ is stochastically bounded, we have (Theorem 2.5, de Acosta (1980)) that

$$\sup_n \mathbb{E}\exp\Big\{\theta\sqrt{\frac{b_n}{n}}\max_{0\le k\le t_n}|\bar S(k)|\Big\} < \infty$$

for all $\theta > 0$. This, combined with (3.9), the fact that

$$\begin{aligned}0 &\le \big\{\max_{0\le k\le t_n}\bar S(k) - \min_{0\le k\le t_n}\bar S(k)\big\} - \#\big\{\bar S[0,\ t_n]\big\}\\ &\le \big\{\max_{0\le k\le t_n}\bar S(k) - \min_{0\le k\le t_n}\bar S(k)\big\} \le 2\max_{0\le k\le t_n}|\bar S(k)|\end{aligned}$$

and the dominated convergence theorem, implies that

$$\mathbb{E}\exp\Big\{\theta\sqrt{\frac{b_n}{n}}\Big(\big\{\max_{0\le k\le t_n}\bar S(k) - \min_{0\le k\le t_n}\bar S(k)\big\} - \#\big\{\bar S[0,\ t_n]\big\}\Big)\Big\} \longrightarrow 1 \qquad (n\to\infty).$$

By (3.8), therefore,

$$\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{E}\exp\Big\{\theta\sqrt{\frac{b_n}{n}}\big\{\max_{0\le k\le \gamma_n t_n}\bar S(k) - \min_{0\le k\le \gamma_n t_n}\bar S(k)\big\} - \#\big\{\bar S[0,\gamma_n t_n]\big\}\Big\} = 0$$

for every $\theta > 0$.

Finally, (3.7) follows from a standard application of Chebyshev's inequality.

4. Proof of Theorem 2.

By the fact that

$$\#\big\{S[0,n]\big\} \le \max_{0\le k\le n} S(k) - \min_{0\le k\le n} S(k) \le 2\max_{0\le k\le n}|S(k)|$$

we have

$$\mathbb{P}\Big\{\#\big\{S[0,n]\big\} \le \lambda\sqrt{\frac{n}{b_n}}\Big\} \ge \mathbb{P}\Big\{\max_{0\le k\le n}|S(k)| \le \frac{\lambda}{2}\sqrt{\frac{n}{b_n}}\Big\}.$$

According to Theorem 4.5 in de Acosta (1983),

$$\liminf_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\Big\{\max_{0\le k\le n}|S(k)|\le\frac{\lambda}{2}\sqrt{\frac{n}{b_n}}\Big\}\ge-\frac{\sigma^2\pi^2}{2\lambda^2}.$$

We thus have

$$\liminf_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\Big\{\#\{S[0,n]\}\le\lambda\sqrt{\frac{n}{b_n}}\Big\}\ge-\frac{\sigma^2\pi^2}{2\lambda^2}. \tag{4.1}$$

On the other hand, let $t>0$ be arbitrary but fixed and write $t_n=[tn/b_n]$ and $\gamma_n=[n/t_n]$. We have

$$\begin{aligned}&\mathbb{P}\Big\{\#\{S[0,n]\}\le\lambda\sqrt{\frac{n}{b_n}}\Big\}\le\mathbb{P}\Big\{\#\{S[0,\gamma_n t_n]\}\le\lambda\sqrt{\frac{n}{b_n}}\Big\}\\&\le\mathbb{P}\Big\{\max_{1\le i\le\gamma_n}\#\{S[(i-1)t_n,it_n]\}\le\lambda\sqrt{\frac{n}{b_n}}\Big\}=\Big(\mathbb{P}\Big\{\#\{S[0,t_n]\}\le\lambda\sqrt{\frac{n}{b_n}}\Big\}\Big)^{\gamma_n}.\end{aligned}$$

From (2.4) we have

$$\mathbb{P}\Big\{\#\{S[0,t_n]\}\le\lambda\sqrt{\frac{n}{b_n}}\Big\}\longrightarrow\mathbb{P}\Big\{\max_{0\le s\le1}W(s)-\min_{0\le s\le1}W(s)\le\frac{\lambda}{\sqrt{t}}\Big\}.$$

Consequently,

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\Big\{\#\{S[0,n]\}\le\lambda\sqrt{\frac{n}{b_n}}\Big\}\le\frac{1}{t}\log\mathbb{P}\Big\{\max_{0\le s\le1}W(s)-\min_{0\le s\le1}W(s)\le\frac{\lambda}{\sqrt{t}}\Big\}. \tag{4.2}$$

The exact distribution of the Brownian range was first found in Feller (1951), which gives that

$$\mathbb{P}\Big\{\max_{0\le s\le1}W(s)-\min_{0\le s\le1}W(s)\le\frac{\lambda}{\sqrt{t}}\Big\}\sim\frac{8\sigma^2t}{\lambda^2}\exp\Big\{-\frac{\pi^2\sigma^2}{2\lambda^2}t\Big\}\qquad(t\to\infty).$$

Letting $t\longrightarrow\infty$ on the right hand side of (4.2) gives

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\Big\{\#\{S[0,n]\}\le\lambda\sqrt{\frac{n}{b_n}}\Big\}\le-\frac{\pi^2\sigma^2}{2\lambda^2}. \tag{4.3}$$

Finally, Theorem 2 follows from (4.1) and (4.3).

5. Proof of Theorem 3.

Let $\theta>1$ be fixed and write $n_k=[\theta^k]$ $(k=1,2,\cdots)$. Taking $b_n=\log\log n$ in Theorem 2 gives that

$$\sum_k\mathbb{P}\Big\{\#\{S[0,n_k]\}\le\lambda\sqrt{\frac{n_k}{\log\log n_k}}\Big\}<\infty$$

for any $\lambda<\pi\sigma/\sqrt{2}$. By Borel-Cantelli lemma,

$$\liminf_{k\to\infty}\sqrt{\frac{\log\log n_k}{n_k}}\#\{S[0,n_k]\}\ge\frac{\pi}{\sqrt{2}}\sigma\qquad a.s.$$

For any $n_k \le n \le n_{k+1}$,

$$\sqrt{\frac{\log\log n}{n}}\#\big\{S[0,n]\big\} \ge \sqrt{\frac{\log\log n_{k+1}}{n_{k+1}}}\#\big\{S[0,n_k]\big\} \ge \big(1+o(1)\big)\theta^{-1/2}\sqrt{\frac{\log\log n_k}{n_k}}\#\big\{S[0,n_k]\big\}.$$

So we have

$$\liminf_{n\to\infty}\sqrt{\frac{\log\log n}{n}}\#\big\{S[0,n]\big\} \ge \theta^{-1/2}\frac{\pi}{\sqrt{2}}\sigma \qquad a.s.$$

Letting $\theta \to 1^+$ gives that

$$\liminf_{n\to\infty}\sqrt{\frac{\log\log n}{n}}\#\big\{S[0,n]\big\} \ge \frac{\pi}{\sqrt{2}}\sigma \qquad a.s. \tag{5.1}$$

Similarly, by Theorem 1 and Borel-Cantelli lemma we can prove that

$$\limsup_{n\to\infty}\frac{1}{\sqrt{n\log\log n}}\#\big\{S_1[0,n]\cap\cdots\cap S_p[0,n]\big\} \le \sqrt{\frac{2}{p}}\sigma \qquad a.s. \tag{5.2}$$

On the other hand, write $m_k = k^k$ $(k = 1, 2, \cdots)$. By Theorem 2 we have that for any $\lambda > \pi\sigma/\sqrt{2}$

$$\begin{aligned}&\sum_k \mathbb{P}\Big\{\#\big\{S[m_k, m_{k+1}]\big\} \le \lambda\sqrt{\frac{m_{k+1}}{\log\log m_{k+1}}}\Big\}\\ &= \sum_k \mathbb{P}\Big\{\#\big\{S[0, m_{k+1}-m_k]\big\} \le \lambda\sqrt{\frac{m_{k+1}}{\log\log m_{k+1}}}\Big\} = \infty.\end{aligned}$$

Notice that $\Big\{\#\big\{S[m_k, m_{k+1}]\big\}\Big\}_k$ is an independent sequence. By Borel-Cantelli lemma,

$$\liminf_{k\to\infty}\sqrt{\frac{\log\log m_{k+1}}{m_{k+1}}}\#\big\{S[m_k, m_{k+1}]\big\} \le \frac{\pi}{\sqrt{2}}\sigma \qquad a.s.$$

Consider the inequality,

$$\#\big\{S[0, m_{k+1}]\big\} \le \#\big\{S[m_k, m_{k+1}]\big\} + \#\big\{S[0, m_k]\big\}.$$

Since

$$\sqrt{m_k\log\log m_k} = o\Big(\sqrt{\frac{m_{k+1}}{\log\log m_{k+1}}}\Big) \qquad (k\to\infty)$$

by (5.2) we have

$$\limsup_{k\to\infty}\sqrt{\frac{\log\log m_{k+1}}{m_{k+1}}}\#\big\{S[0, m_k]\big\} = 0 \qquad a.s.$$

Hence,

$$\liminf_{k\to\infty}\sqrt{\frac{\log\log m_{k+1}}{m_{k+1}}}\#\big\{S[0, m_{k+1}]\big\} \le \frac{\pi}{\sqrt{2}}\sigma \qquad a.s.$$

Consequently,

$$\liminf_{n\to\infty}\sqrt{\frac{\log\log n}{n}}\#\big\{S[0,n]\big\}\le\frac{\pi}{\sqrt{2}}\sigma\qquad a.s. \tag{5.3}$$

Hence, (1.7) follows from (5.1) and (5.3).

We now come to the proof of the lower bound of (1.6). That is, we prove

$$\limsup_{n\to\infty}\frac{1}{\sqrt{n\log\log n}}\#\big\{S_1[0,n]\cap\cdots\cap S_p[0,n]\big\}\ge\sqrt{\frac{2}{p}}\sigma\qquad a.s. \tag{5.4}$$

As $p=1$, the proof of Theorem can be carried out in a way similar to the one for (5.3). The case $p>1$ is more delicate for lack of subadditivity.

For given $\bar{x}=(x_1,\cdots,x_p)\in(\mathbb{Z})^p$, we introduce the notation $\mathbb{P}^{\bar{x}}$ for the probability induced by the random walks $S_1(n),\cdots,S_p(n)$ in the case when $S_1(n),\cdots,S_p(n)$ start at $x_1,\cdots,x_p$, respectively. The notation $\mathbb{E}^{\bar{x}}$ denote the expectation correspondent to $\mathbb{P}^{\bar{x}}$. To be consistent with the notation we used before, we have $\mathbb{P}^{(0,\cdots,0)}=\mathbb{P}$ and $\mathbb{E}^{(0,\cdots,0)}=\mathbb{E}$. Write

$$||\bar{x}||=\max_{1\le j\le p}|x_j|\qquad \bar{x}=(x_1,\cdots,x_p)\in(\mathbb{R})^p.$$

By the argument used in the proof of (5.6), Chen (2004), we will have (5.4) if

$$\liminf_{n\to\infty}\frac{1}{b_n}\log\inf_{||\bar{x}||\le\sqrt{n/b_n}}\mathbb{P}^{\bar{x}}\Big\{\#\big\{S_1[0,n]\cap\cdots\cap S_p[0,n]\big\}\ge\lambda\sqrt{nb_n}\Big\}\ge-\frac{p\lambda^2}{2\sigma^2}. \tag{5.5}$$

for any $\lambda>0$.

Take $t_n=[n/b_n]$ and $T_x=\inf\{n\ge 0;\quad S(n)=x\}$. Then for any $\bar{x}\in\mathbb{Z}^p$ with $||\bar{x}||\le\sqrt{n/b_n}$,

$$\begin{aligned}
&\mathbb{P}^{\bar{x}}\Big\{\#\big\{S_1[0,n]\cap\cdots\cap S_p[0,n]\big\}\ge\lambda\sqrt{nb_n}\Big\}\\
&=\mathbb{P}\Big\{\#\Big\{\bigcap_{j=1}^p\big(x_j+S_j[0,n]\big)\Big\}\ge\lambda\sqrt{nb_n}\Big\}\\
&\ge\sum_{k_1,\cdots,k_p=0}^{t_n}\mathbb{P}\Big\{T_{-x_1}=k_1,\cdots,T_{-x_p}=k_p;\ \#\Big\{\bigcap_{j=1}^p\big(x_j+S_j[k_j,n]\big)\Big\}\ge\lambda\sqrt{nb_n}\Big\}\\
&=\sum_{k_1,\cdots,k_p=0}^{t_n}\Big(\prod_{j=1}^p\mathbb{P}\{T_{-x_j}=k_j\}\Big)\mathbb{P}\Big\{\#\big\{S_1[0,n-k_1]\cap\cdots\cap S_p[0,n-k_p]\big\}\ge\lambda\sqrt{nb_n}\Big\}\\
&\ge\Big(\inf_{|x|\le\sqrt{n/b_n}}\mathbb{P}\{T_x\le t_n\}\Big)^p\mathbb{P}\Big\{\#\big\{S_1[0,n-t_n]\cap\cdots\cap S_p[0,n-t_n]\big\}\ge\lambda\sqrt{nb_n}\Big\}.
\end{aligned}$$

By Lemma 5.1 given below, there is a constant $\delta>0$ such that

$$\begin{aligned}
&\inf_{||\bar{x}||\le\sqrt{n/b_n}}\mathbb{P}^{\bar{x}}\Big\{\#\big\{S_1[0,n]\cap\cdots\cap S_p[0,n]\big\}\ge\lambda\sqrt{nb_n}\Big\}\\
&\ge\delta\mathbb{P}\Big\{\#\big\{S_1[0,n-t_n]\cap\cdots\cap S_p[0,n-t_n]\big\}\ge\lambda\sqrt{nb_n}\Big\}
\end{aligned}$$

for all $n\ge 1$. Hence, (5.5) follows from Theorem 1 with n being replaced by $n-t_n$.

We end this section with the following lemma.

Lemma 5.1. *For any $x \in \mathbb{Z}$, write*

$$T_x = \inf\{n \geq 0; \quad S(n) = x\}.$$

Then

$$\liminf_{n\to\infty} \inf_{|x|\leq C\sqrt{n}} \mathbb{P}\{T_x \leq n\} > 0 \tag{5.6}$$

for any $C > 0$ and

$$\lim_{\epsilon\to 0^+} \liminf_{n\to\infty} \inf_{|x|\leq \epsilon\sqrt{n}} \mathbb{P}\{T_x \leq n\} = 1. \tag{5.7}$$

Proof. Recall that the random walk $\{S(n)\}$ is said to be aperiodic if the greatest common factor of the set

$$\Big\{n \geq 1; \quad \mathbb{P}\{S(n) = 0\} > 0\Big\}$$

is 1. We first prove Lemma 5.1 under the assumption of aperiodicity.

By Markov property,

$$\mathbb{P}\{S(k) = x\} = \sum_{j=0}^{k} \mathbb{P}\{T_x = j,\ S(k) = x\} = \sum_{j=0}^{k} \mathbb{P}\{T_x = j\}\mathbb{P}\{S(k-j) = 0\}.$$

Summing up on the both sides,

$$\begin{aligned}
&\sum_{k=0}^{n} \mathbb{P}\{S(k) = x\} = \sum_{k=0}^{n}\sum_{j=0}^{k} \mathbb{P}\{T_x = j\}\mathbb{P}\{S(k-j) = 0\} \\
&= \sum_{j=0}^{n} \mathbb{P}\{T_x = j\} \sum_{k=j}^{n} \mathbb{P}\{S(k-j) = 0\} \leq \mathbb{P}\{T_x \leq n\} \sum_{k=0}^{n} \mathbb{P}\{S(k) = 0\}.
\end{aligned}$$

By the Remark in p. 661 of Le Gall and Rosen (1991) the aperiodicity of the random walk implies

$$\sup_{x\in\mathbb{Z}} \Big|\sqrt{n}\mathbb{P}\{S(n) = x\} - \frac{1}{\sqrt{2\pi}\sigma} \exp\Big\{-\frac{x^2}{2n\sigma^2}\Big\}\Big| \longrightarrow 0 \qquad (n\to\infty).$$

which gives (5.6) and (5.7).

We now prove (5.6) and (5.7) without assuming aperiodicity. Let $0 < \eta < 1$ be fixed and let $\{\delta_n\}_{n\geq 1}$ be i.i.d. Bernoulli random variables with the common law:

$$\mathbb{P}\{\delta_1 = 0\} = 1 - \mathbb{P}\{\delta_1 = 1\} = \eta.$$

We assume independence between $\{S(n)\}$ and $\{\delta_n\}$.

Define the renewal sequence $\{\tau_k\}_{k\geq 0}$ by

$$\tau_0 = 0 \quad \text{and} \quad \tau_{k+1} = \inf\{n > \tau_k;\ \delta_n = 1\}.$$

Then $\{\tau_k - \tau_{k-1}\}_{k\geq 1}$ is an i.i.d. sequence with common distribution

$$\mathbb{P}\{\tau_1 = n\} = (1-\eta)\eta^{n-1} \qquad n = 1, 2, \cdots.$$

Consider the random walk $\tilde{S}(n) = S(\tau_n)$. By the fact that

$$\mathbb{P}\{S(\tau_1) = 0\} = (1-\eta)\sum_{k=1}^{\infty}\eta^{k-1}\mathbb{P}\{S(k) = 0\} > 0,$$

$\{\tilde{S}(n)\}$ is aperiodic. Applying what we have proved to $\{\tilde{S}(n)\}$,

$$\liminf_{n\to\infty}\inf_{|x|\leq C\sqrt{n}}\mathbb{P}\{\tilde{T}_x \leq n\} > 0 \quad \text{and} \quad \lim_{\epsilon\to 0^+}\liminf_{n\to\infty}\inf_{|x|\leq \epsilon\sqrt{n}}\mathbb{P}\{\tilde{T}_x \leq n\} = 1 \tag{5.8}$$

where

$$\tilde{T}_x = \inf\{n \geq 0; \quad S(\tau_n) = x\}.$$

Notice that

$$\tau_{\tilde{T}_x} \geq T_x \qquad x \in \mathbb{Z}.$$

Take $0 < \theta < 1$ such that $\theta(1-\eta)^{-1} < 1$. Then for any $x \in \mathbb{Z}$,

$$\mathbb{P}\{T_x > n\} \leq \mathbb{P}\{\tau_{\tilde{T}_x} > n\} \leq \mathbb{P}\{\max_{k\leq\theta n}\tau_k > n\} + \mathbb{P}\{\tilde{T}_x > \theta n\}.$$

By the classic law of large numbers, the first term on the right tends to zero as $n\to\infty$. Therefore, (5.6) and (5.7) follows from (5.8) with n being replaced by $[\theta n]$. $\square$

References

[1] Bass, R. F. and Chen, X. (2004). Self-intersection local time: critical exponent and laws of the iterated logarithm. *Ann. Probab.* **32** 3221-3247.

[2] Bass, R. F., Chen, X. and Rosen, J. (2005). Large deviations for renormalized self-intersection local times of stable processes. *Ann. Probab.* (to appear).

[3] Bass, R. F., Chen, X. and Rosen, J. (2005). Moderate deviations for the range and self-intersections of planar random walks. (in preparation).

[4] Bass, R. F. and Kumagai, T. (2002). Laws of the iterated logarithm for the range of random walks in two and three dimensions. *Ann. Probab.* **30** 1369-1396.

[5] Chen, X. (2004). Exponential asymptotics and law of the iterated logarithm for intersection local times of random walks. *Ann. Probab.* **32** 3248-3300.

[6] Chen, X. (2005). Moderate deviations and law of the iterated logarithm for intersections of the ranges of random walks. *Ann. Probab.* (to appear).

[7] Chen, X. and Li, W. (2004). Large and moderate deviations for intersection local times. *Probab Theor. Rel. Fields* **128** 213-254.

[8] Chen, X., Li, W. and Rosen, J. (2004). Large deviations for local times of stable processes and stable random walks in 1 dimension. *Electron. J. Probab.* (to appear).

[9] Chen, X. and Rosen, J. (2005). Exponential asymptotics for intersection local times of stable processes and random walks. *Annales de l'Institut Henri Poincare* (to appear).

[10] de Acosta, A. (1980). Exponential moments of vector valued random series and triangular arrays. *Ann. Probab.* **8** 381-389.

[11] de Acosta, A. (1983). Small deviations in the functional central limit theorem with application to functional laws of the iterated logarithm. *Ann. Probab.* **11** 78-101.

[12] Dembo, A. and Zeitouni, O. (1998). *Large Deviations Techniques and Applications. (2nd ed.)* Springer, New York.

[13] Dvoretzky, A., Erdös, P. and Kakutani, S. (1950). Double points of paths of Brownian motions in n-space. *Acta Sci. Math. Szeged* **12**, Leopoldo, Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 75-81.

[14] Dvoretzky, A., Erdös, P. and Kakutani, S. (1954). Multiple points of paths of Brownian motion in the plane. *Bull. Res. Council Israel.* **3**, 364-371.

[15] Feller, W. (1951). The asymptotic distribution of the range of sums of independent random variables. *Ann. Math. Stat.* **22** 427-432.

[16] Hamana, Y. (1998). An almost sure invariance principle for the range of random walks. *Stochastic Process. Appl.* **78** 131-143.

[17] Jain, N. C. and Pruitt, W. E. (1971). The range of transient random walks. *J. Anal. Math.* **24** 369-393.

[18] Jain, N. C. and Pruitt, W. E. (1972). The law of the iterated logarithm for the range of random walk. *Ann. Math. Statist.* **43** 1692-1697.

[19] Jain, N. C. and Pruitt, W. E. (1974). Further limit theorems for the range of random walk. *J. Anal. Math.* **27** 94-117.

[20] Klenke, A. and Mörters, P. (2004).The multifractal spectrum of Brownian intersection local times. *Ann. Probab.* (to appear).

[21] Le Gall, J-F. (1986a). Propriétés d'intersection des marches aléatoires. I. Convergence vers le temps local d'intersection. *Comm. Math. Phys.* **104**, 471-507.

[22] Le Gall, J-F. (1986b). Propriétés d'intersection des marches aléatoires. II. Étude des cas critiques. *Comm. Math. Phys.* **104** 509-528.

[23] Le Gall, J-F. and Rosen, J. (1991). The range of stable random walks. Ann. Probab. **19**, 650–705.

[24] Marcus, M. B. and Rosen, J. (1997) Laws of the iterated logarithm for intersections of random walks on Z^4. *Ann. Inst. H. Poincaré Probab. Statist.* **33**, 37–63.

[25] Rosen, J. (1997). Laws of the iterated logarithm for triple intersections of three-dimensional random walks. *Electron. J. Probab.* **2**, 1-32.