Spatial Brownian motion in renormalized Poisson potential: A critical case

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Abstract

Let $B_s$ be a three dimensional Brownian motion and $\omega(dx)$ be an independent Poisson field on $\mathbb{R}^3$. It is claimed that for any $t > 0$,

$$
E_0 \exp \left\{ \theta \int_0^t \nabla (B_s) ds \right\} \begin{cases} < \infty & \text{a.s. when } \theta < \frac{1}{16} \\
= \infty & \text{a.s. when } \theta > \frac{1}{16} 
\end{cases}
$$

conditioning on the renormalized Poisson potential

$$
\nabla (x) = \int_{\mathbb{R}^3} \frac{1}{|y - x|^2} [\omega(dy) - dy].
$$

Under $\theta < 1/16$, the long term behaviors of the above quenched exponential moments are investigated and the theorems are given in the forms of integral test. The main results of this paper are mathematically relevant to the Hardy’s inequality

$$
\int_{\mathbb{R}^3} \frac{f^2(x)}{|x|^2} dx \leq 4 \| \nabla f \|_2^2 \quad f \in W^{1,2}(\mathbb{R}^3)
$$

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1 Introduction

Consider a particle doing a random movement in the space $\mathbb{R}^d$ (Very soon, we will focus on the case $d = 3$). The trajectory of the particle is described by a $d$-dimensional Brownian motion $B_s$. Independently, there are a family of the obstacles randomly located in the space $\mathbb{R}^d$. Assume that each obstacle has mass 1 and that the obstacles are distributed in $\mathbb{R}^d$ according to a Poisson field $\omega(dx)$ with the Lebesgue measure $dx$ as its intensity measure. Throughout, the notation “$\mathbb{P}$” and “$\mathbb{E}$” are used for the probability law and the expectation, respectively, generated by the Poisson field $\omega(dx)$, while the notations “$\mathbb{P}_x$” and “$\mathbb{E}_x$” are for the probability law and the expectation, respectively, of the Brownian motion $B_s$ with $B_0 = x$.

In the recent papers [7] and [6], the renormalized random potential

$$\nabla(x) = \int_{\mathbb{R}^d} \frac{1}{|y - x|^p} [\omega(dy) - dy] \quad x \in \mathbb{R}^d$$

are introduced and the correspondent quenched exponential moments

$$\mathbb{E}_0 \exp \left\{ \pm \theta \int_0^t \nabla(B_s) ds \right\}$$

are studied for their roles in the models of the Brownian motions in Poisson obstacles and in the parabolic Anderson models. By Corollary 1.3 in [7], the potential $\nabla$ is well defined as a random integral if and only if $d/2 < p < d$. In this case the time-integral

$$\int_0^t \nabla(B_s) ds$$

is well defined and satisfies the integrabilities

$$\mathbb{E} \otimes \mathbb{E}_0 \exp \left\{ - \theta \int_0^t \nabla(B_s) ds \right\} < \infty \quad \text{and} \quad E_0 \exp \left\{ - \theta \int_0^t \nabla(B_s) ds \right\} < \infty \quad \text{a.s.}$$

for every $\theta > 0$ and $t > 0$.

The study of exponential moment with positive coefficient is far more delicate. By Theorem 1.4 in [7],

$$\mathbb{E} \otimes \mathbb{E}_0 \exp \left\{ \theta \int_0^t \nabla(B_s) ds \right\} = \infty$$

for every $\theta > 0$ and $t > 0$. As for the quenched integrability, it has been shown (Theorem 1.5, [7]) that for any $\theta > 0$ and $t > 0$, with probability 1

$$\mathbb{E}_0 \exp \left\{ \theta \int_0^t \nabla(B_s) ds \right\} \begin{cases} < \infty & \text{a.s. when } p < 2 \\ = \infty & \text{a.s. when } p > 2 \end{cases}$$
Under $p < 2$, the author ([6]) recently observed that

$$
\lim_{t \to \infty} \frac{1}{t} \left( \frac{\log \log t}{\log t} \right)^{2-p} \log \mathbb{E}_0 \exp \left\{ \theta \int_0^t \nabla (B_s) ds \right\}
$$

$$
= \frac{1}{2} p^{\frac{p}{2}} (2 - p)^{\frac{1}{2-p}} \left( \frac{d \sigma(d,p)}{d-p} \frac{2}{2 + d - p} \right)^{\frac{2}{2-p}} \text{ a.s. - } \mathbb{P}
$$

where $\sigma(d,p) > 0$ is the best constant of the inequality

$$
\int_{\mathbb{R}^d} \frac{f^2(x)}{|x|^p} dx \leq C \|f\|_2^{2-p} \|\nabla f\|_2^p \quad f \in W^{1,2}(\mathbb{R}^d)
$$

This paper is to investigate the quenched exponential moment

$$
\mathbb{E}_0 \exp \left\{ \theta \int_0^t \nabla (B_s) ds \right\} \quad \theta > 0, \quad t > 0
$$

in the critical case $p = 2$, in which the constraint $d/2 < p < d$ leads to $d = 3$. From now on, $B_s$ is a three-dimensional Brownian motion and the random potential $\nabla(x)$ on $\mathbb{R}^3$ is defined as

$$
\nabla(x) = \int_{\mathbb{R}^3} \frac{1}{|y - x|^2} \left[ \omega(dy) - dy \right] \quad x \in \mathbb{R}^3
$$

Concerning integrability, it has been pointed out (Theorem 1.5, [7]) that when $d = 3$ and $p = 2$,

$$
\mathbb{E}_0 \exp \left\{ \theta \int_0^t \nabla (B_s) ds \right\} = \infty \quad \text{a.s.}
$$

for large $\theta > 0$ and for every $t > 0$. The following theorem indicates a phase transition.

**Theorem 1.1** For every $t > 0$,

$$
\mathbb{E}_0 \exp \left\{ \theta \int_0^t \nabla (B_s) ds \right\} \begin{cases}
< \infty & \text{a.s. when } \theta \leq \frac{1}{16} \\
= \infty & \text{a.s. when } \theta > \frac{1}{16}
\end{cases}
$$

Theorem 1.1 applies to the parabolic Anderson equation

$$
\begin{cases}
\partial_t u(t, x) = \kappa \Delta u(t, x) + \theta \nabla(x) u(t, x) \\
u(0, x) = 1
\end{cases}
$$

where $\kappa > 0$ is a constant called diffusion coefficient. Indeed, consider the time-space field

$$
u_{\theta}(t, x) = \mathbb{E}_x \exp \left\{ \theta \int_0^t \nabla (B_{2ns}) ds \right\} = \mathbb{E}_x \exp \left\{ \frac{\theta}{2\kappa} \int_0^{2nt} \nabla (B_s) ds \right\}$$
By translation invariance of the Poisson field, for any $x \in \mathbb{R}^d$

$$\left\{ u_\theta(t, x); \ t \geq 0 \right\} \overset{d}{=} \left\{ u_\theta(t, 0); \ t \geq 0 \right\}$$

By Theorem 1.1, $u_\theta(t, x) < \infty$ a.s. for every $x \in \mathbb{R}^d$ and $t > 0$ when $\theta < \kappa/8$. The same argument as the one for Proposition 1.6, [7] concludes that $u_\theta(t, x)$ is a mild solution to the equation (1) when $\theta < \kappa/8$.

In view of the limit law (1) obtained in the non-critical case, a natural problem is the asymptotic behaviors in the critical case. Recall that a positive function $\gamma(t)$ on $\mathbb{R}^+$ is said to be regularly varying at $\infty$ if the limit

$$\lim_{t \to \infty} \frac{\gamma(2t)}{\gamma(t)} = c(\gamma)$$

exists. A regular-varying (at $\infty$) function $\gamma(t)$ is said to be slowly varying at $\infty$, if $c(\gamma) = 1$. It is a classic fact that for each function $\gamma(t)$ regularly varying at $\infty$, it yields a representation $\gamma(t) = t^\beta l(t)$, where $\beta$ is a constant and $l(t)$ is a slow-varying function.

Throughout, $l(t)$ is a slow-varying function.

**Theorem 1.2** Under $0 < \theta < 16^{-1}$,

$$\limsup_{t \to \infty} t^{\frac{k+1}{k-1}} l(t) \frac{\log \mathbb{E}_0 \exp \left\{ \theta \int_0^t \nabla(B_s) ds \right\}}{\frac{\log \mathbb{E}_0 \exp \left\{ \theta \int_0^t \nabla(B_s) ds \right\}}{\pi^{(k-1)/2}}} = \begin{cases} 
0 & \text{a.s. if } \int_1^\infty \frac{dt}{t \cdot l(t)} < \infty \\
\infty & \text{a.s. if } \int_1^\infty \frac{dt}{t \cdot l(t)} = \infty
\end{cases}$$

where $k = \left[(8\theta)^{-1}\right]$ is the integer part of $(8\theta)^{-1}$.

**Theorem 1.3** Under $0 < \theta < 16^{-1}$,

$$\liminf_{t \to \infty} t^{\frac{k+1}{k-1}} l(t) \frac{\log \mathbb{E}_0 \exp \left\{ \theta \int_0^t \nabla(B_s) ds \right\}}{\frac{\log \mathbb{E}_0 \exp \left\{ \theta \int_0^t \nabla(B_s) ds \right\}}{\pi^{(k-1)/2}}} = \begin{cases} 
0 & \text{a.s. if } \int_1^\infty \frac{1}{t} \exp \left\{ -c \cdot l(t) \right\} dt = \infty \text{ for some } c > 0 \\
\infty & \text{a.s. if } \int_1^\infty \frac{1}{t} \exp \left\{ -c \cdot l(t) \right\} dt < \infty \text{ for every } c > 0
\end{cases}$$

where $k = \left[(8\theta)^{-1}\right]$ is the integer part of $(8\theta)^{-1}$.
Our theorems present both $\theta$-dependence and $\theta$-independence. On the one hand, putting $\theta$ into different sub-intervals of the partition

$$
(0, \frac{1}{16}) = (\frac{1}{24}, \frac{1}{16}) \cup \bigcup_{k=3}^{\infty} \left( \frac{1}{8(k+1)}, \frac{1}{8k} \right)
$$

leads to drastically different asymptotic behaviors. On the other hand, moving $\theta$ around within the same sub-interval does not bring any change to the behavior of the system.

Our main results indicates the as far as the strong limit is concerned, there is not “right” deterministic normalization to our system. Indeed, by Theorem 1.2 and Theorem 1.3, for any $0 < \theta < \frac{1}{16}$, and for any positive deterministic function $\gamma(t)$ regularly varying at $\infty$, with probability 1

$$
\limsup_{t \to \infty} \gamma(t)^{-1} \log E_0 \exp \left\{ \theta \int_0^t V(B_s) ds \right\} = 0 \text{ or } \infty
$$

$$
\liminf_{t \to \infty} \gamma(t)^{-1} \log E_0 \exp \left\{ \theta \int_0^t V(B_s) ds \right\} = 0 \text{ or } \infty
$$

This pattern sharply contrasts (1) observed in the non-critical setting. Letting $l(t)$ be some specific functions, we get the following results:

$$
\limsup_{n \to \infty} t^{\frac{k+1}{k-1}} (\log t)^{-\frac{2}{3(k-1)}} \log E_0 \exp \left\{ \theta \int_0^t V(B_s) ds \right\} = \infty \quad \text{a.s.}
$$

On the other hand,

$$
\limsup_{n \to \infty} t^{\frac{k+1}{k-1}} ((\log t)(\log \log t)^{1+\delta})^{-\frac{2}{3(k-1)}} \log E_0 \exp \left\{ \theta \int_0^t V(B_s) ds \right\} = 0 \quad \text{a.s.}
$$

As for the liminf behavior,

$$
\liminf_{n \to \infty} t^{\frac{k+1}{k-1}} (\log \log t)^{\frac{2}{3(k-1)}} \log E_0 \exp \left\{ \theta \int_0^t V(B_s) ds \right\} = 0 \quad \text{a.s.}
$$

On the other hand, for any $l(t) \gg \log \log t$ as $t \to \infty$,

$$
\liminf_{n \to \infty} t^{\frac{k+1}{k-1}} l(t)^{\frac{2}{3(k-1)}} \log E_0 \exp \left\{ \theta \int_0^t V(B_s) ds \right\} = \infty \quad \text{a.s.}
$$

Given the non-deterministic asymptotic behaviors observed from Theorem 1.2 and Theorem 1.3, the weak law (if any) becomes an interesting problem. In view of Theorem 1.2 and Theorem 1.3, one might expect that the process

$$
t^{-\frac{k+1}{k-1}} \log E_0 \exp \left\{ \theta \int_0^t V(B_s) ds \right\}
$$
converges to a non-degenerated distribution. We leave this problem to future study.

The rest of the paper is organized as following. In section 2, we develop key tools for the estimations needed in later sections. Some of these tools are interesting for their own sake. Among them are the estimation by chaining (Lemma 2.1), the Slepian-type domination for infinitely divisible fields (Lemma 2.3), certain isoperimetric inequalities (Lemmas 2.5 and 2.7). In section 3, we establish the lower bounds for our main theorems. The main efforts is to associate the problem with Hardy inequality through the Feynman-Kac representation. In section 4, we deal with the upper bounds for the main theorems, which appears to be a much more delicate matters. The biggest challenge is on the localization of Brownian paths.

2 Basic estimates

In this section we provide some basic results that will be used in our proofs. We state them separately for a convenient reference.

2.1 Truncating Poisson potentials

To reserve continuity we adopt a smooth truncation to the shape function. Let the smooth function $\alpha: \mathbb{R}^+ \to [0, 1]$ satisfy the following properties: $\alpha(\lambda) = 1$ on $[0, 1]$, $\alpha(\lambda) = 0$ for $\lambda \geq 3$ and $-1 \leq \alpha'(\lambda) \leq 0$.

Let $a > 0$ be fixed but arbitrary. Write

$$L_a(x) = \frac{1 - \alpha(a^{-1}|x|)}{|x|^2} \quad \text{and} \quad V_{a,\epsilon}(x) = \int_{\mathbb{R}^3} L_a(y - x) [\omega(\epsilon dx) - \epsilon dx]$$

Throughout this section, $D \subset \mathbb{R}^3$ is a fixed bounded set.

**Lemma 2.1** For any $\theta > 0$ and fixed $a > 0$

$$\mathbb{E} \exp \left\{ \theta \sup_{x \in D} |V_{a,1}(x)| \right\} < \infty$$

Further, for given $\theta > 0$ one can take $a > 0$ large enough so

$$\sup_{0<\epsilon<1} \mathbb{E} \exp \left\{ \theta (\log \epsilon^{-1}) \sup_{x \in D} |V_{a,\epsilon}(x)| \right\} < \infty$$

**Proof:** Due to similarity, we only prove (2.1). Write

$$\Psi(\lambda) = e^\lambda - 1 - \lambda \quad \lambda \in \mathbb{R}$$
We have
\[
\mathbb{E} \exp \left\{ \pm \theta (\log \epsilon^{-1}) \nabla_{a,\epsilon}(0) \right\} = \exp \left\{ \epsilon \int_{\mathbb{R}^3} \Psi \left( \pm \theta (\log \epsilon^{-1}) \frac{1 - \alpha(a^{-1}|x|)}{|x|^2} \right) dx \right\} \\
\leq \exp \left\{ \epsilon \int_{\mathbb{R}^3} \Psi \left( \theta (\log \epsilon^{-1}) \frac{1 - \alpha(a^{-1}|x|)}{|x|^2} \right) dx \right\}
\]
where the second step follows from the fact that \( \Psi(\lambda) \leq \Psi(\lambda) \) for any \( \lambda > 0 \). Therefore,
\[
\mathbb{E} \exp \left\{ \theta (\log \epsilon^{-1}) |\nabla_{a,\epsilon}(0)| \right\} \leq 2 \exp \left\{ \epsilon \int_{\mathbb{R}^3} \Psi \left( \theta (\log \epsilon^{-1}) \frac{1 - \alpha(a^{-1}|x|)}{|x|^2} \right) dx \right\}
\]
By integration substitution,
\[
\int_{\mathbb{R}^3} \Psi \left( \theta (\log \epsilon^{-1}) \frac{1 - \alpha(a^{-1}|x|)}{|x|^2} \right) dx \\
= (\log \epsilon^{-1})^{3/2} \int_{\mathbb{R}^3} \Psi \left( \frac{1 - \alpha(a^{-1}|x|\sqrt{\log \epsilon^{-1}})}{|x|^2} \right) dx \\
\leq (\log \epsilon^{-1})^{3/2} \int_{\{ |x| \geq a(\log \epsilon^{-1})^{-1/2} \}} \Psi \left( \frac{\theta}{|x|^2} \right) dx \\
= (\log \epsilon^{-1})^{3/2} \left\{ \int_{\{ a(\log \epsilon^{-1})^{-1/2} \leq |x| \leq 1 \}} \Psi \left( \frac{\theta}{|x|^2} \right) dx + \int_{\{|x| \geq 1\}} \right\} \Psi \left( \frac{\theta}{|x|^2} \right) dx \\
= (\log \epsilon^{-1})^{3/2} \left\{ C \exp \left\{ \theta a^{-2} \log \epsilon^{-1} \right\} + \int_{\{|x| \geq 1\}} \Psi \left( \frac{\theta}{|x|^2} \right) dx \right\}
\]
where the last step follows from the fact that \( \Psi(\lambda) \leq e^\lambda (\lambda > 0) \).
Notice that
\[
\int_{\{|x| \geq 1\}} \Psi \left( \frac{\theta}{|x|^2} \right) dx < \infty
\]
Hence,
\[
\sup_{0 < \epsilon < 1} \mathbb{E} \exp \left\{ \theta (\log \epsilon^{-1}) |\nabla_{a,\epsilon}(0)| \right\} < \infty
\]
when \( \theta < a^2 \).
Similarly, for any \( x, y \in D \) with \( x \neq y \),
\[
\mathbb{E} \exp \left\{ \theta (\log \epsilon^{-1}) \frac{|\nabla_{a,\epsilon}(x) - |\nabla_{a,\epsilon}(y)|}{|x - y|} \right\} \\
\leq 2 \exp \left\{ \epsilon \int_{\mathbb{R}^3} \Psi \left( \frac{\theta \log \epsilon^{-1}}{|x - y|} |L_a(z - x) - L_a(z - y)| \right) dz \right\}
\]
By mean-value theorem,

\[ |L_a(z - x) - L_a(z - y)| \leq \frac{Ca^{-1}}{|z|^2}1_{\{|x|\geq C^{-1}a\}}|x - y| \]

So we have

\[
\int_{\mathbb{R}^3} \Psi\left(\frac{\theta \log \epsilon^{-1} |L_a(z - x) - L_a(z - y)|}{|x - y|}\right) dx \\
\leq \int_{\{|z|\geq C^{-1}a\}} \Psi\left(\frac{\theta Ca^{-1} \log \epsilon^{-1}}{|z|^2}\right) dz \\
= (\log \epsilon^{-1})^{3/2} \int_{\{|z|\geq C^{-1}a(\log \epsilon^{-1})^{-1/2}\}} \Psi\left(\frac{\theta Ca^{-1}}{|z|^2}\right) dz
\]

By an estimate same as the one used in (2.1), therefore,

\[
\sup_{0<\epsilon<1} \sup_{x \neq y} \mathbb{E} \exp\left\{\theta(\log \epsilon^{-1}) \frac{|V_{a,\epsilon}(x) - V_{a,\epsilon}(y)|}{|x - y|}\right\} < \infty
\]

By Theorem D.6, p.313, [5],

\[
\sup_{0<\epsilon<1} \mathbb{E} \exp\left\{\theta(\log \epsilon^{-1}) \sup_{x,y \in D} |V_{a,\epsilon}(x) - V_{a,\epsilon}(y)|\right\} < \infty
\]

So the desired conclusion follows from (2.1) and (2.1). □

Using above lemma, we derive the following almost sure bounds

**Lemma 2.2** For any \( a > 0 \)

\[
\lim_{R \to \infty} (\log R)^{-1} \sup_{|x| \leq R} |V_{a,1}(x)| = 0 \quad a.s.
\]

Further, for any positive sequence \( \epsilon_n \) such that

\[
\limsup_{n \to \infty} \frac{\epsilon_{n+1}}{\epsilon_n} < 1
\]

and any constants \( \beta > 0 \), the strong law

\[
\lim_{n \to \infty} \sup_{|x| \leq \epsilon_n^{-\beta}} |V_{a,\epsilon_n}(x)| = 0 \quad a.s.
\]

holds for sufficiently large \( n \).
**Proof:** The ball $B(0, R)$ is covered by roughly $CR^d$ unit balls. By homogeneity of the field $V(\cdot)$, for each $\delta > 0$

$$\mathbb{P}\left\{ \sup_{|x| \leq R} |V_{a,1}(x)| \geq \delta \log R \right\} \leq CR^d \mathbb{P}\left\{ \sup_{|x| \leq 1} |V_{a,1}(x)| \geq \delta \log R \right\} \leq CR^{-(\theta \delta - d)} \mathbb{E}\exp\left\{ \theta \sup_{|x| \leq 1} |V_{a,1}(x)| \right\}$$

Take $\theta$ sufficiently large so $\theta \delta - d \geq 1$. By (2.1) the exponential moment on the right hand side is finite. Thus,

$$\sum_n \mathbb{P}\left\{ \sup_{|x| \leq 2^n} |V_{a,1}(x)| \geq \delta \log 2^n \right\} < \infty$$

Notice that $\delta > 0$ can be arbitrarily small. By Borel-Cantelli lemma

$$\lim_{n \to \infty} (\log 2^n)^{-1} \sup_{|x| \leq 2^n} |V_{a,1}(x)| = 0 \quad a.s.$$ Hence, (2.2) follows from the fact that $\sup_{|x| \leq R} |V_{a,1}(x)|$ is non-decreasing in $R$.

We now come to the proof of (2.2). First notice that the ball $B(0, \varepsilon_n^{-\beta})$ can be covered by $C \varepsilon_n^{-d\beta}$ balls of radius 1. Thus, for each $u > 0$

$$\mathbb{P}\left\{ \sup_{|x| \leq \varepsilon_n^{-\beta}} |V_{a,\varepsilon_n}(x)| \geq \delta \right\} \leq C \varepsilon_n^{-d\beta} \mathbb{P}\left\{ \sup_{|x| \leq 1} |V_{a,\varepsilon_n}(x)| \geq u \right\} \leq C \varepsilon_n^{\theta \delta - d\beta} \mathbb{E}\exp\left\{ \theta (\log \varepsilon_n^{-1}) \sup_{|x| \leq 1} |V_{a,\varepsilon_n}(x)| \right\}$$

Take $\theta > 0$ sufficiently large so $\theta \delta - d\beta \geq 1$. By (2.1), the exponential moment on the right hand side is bounded uniformly over $n$ when $a > 0$ is sufficiently large. Hence,

$$\sum_n \mathbb{P}\left\{ \sup_{|x| \leq \varepsilon_n^{-\beta}} |V_{a,\varepsilon_n}(x)| \geq \delta \right\} < \infty$$

Therefore, (2.2) follows from Borel-Cantelli lemma. □

### 2.2 Slepian-type domination

Consider a nonnegative random measure $M$ on $\mathbb{R}^d$ taking independent values on disjoint sets such that for every Borel set $A \subset \mathbb{R}^d$

$$\mathbb{E}\exp\{-u M(A)\} = \exp\left\{ -m(A) \int_0^\infty (1 - e^{-ux}) \rho(dx) \right\}, \quad u > 0,$$
where $m$ is a $\sigma$-finite measure on $\mathbb{R}^d$ and $\rho$ is a measure on $(0, \infty)$ with $\int_0^\infty \min\{x, 1\} \rho(dx) < \infty$. $M$ is an example of an infinitely divisible random measure, see [23]. We will call $M$ an infinitely divisible field; it can be viewed as a random distribution of obstacles having random masses. $M$ is a Poisson field if $m(dx) = dx$ and $\rho(dx) = \delta_1(dx)$.

**Lemma 2.3** For every Borel sets $A_1, \ldots, A_n$ and all $c_1, \ldots, c_n \in \mathbb{R}$

$$\mathbb{P}(M(A_1) \leq c_1, \ldots, M(A_n) \leq c_n) \geq \prod_{j=1}^n \mathbb{P}(M(A_j) \leq c_j).$$

This lemma follows from the following more general fact.

**Lemma 2.4** Let $X = (X_1, \ldots, X_n)$ be an infinitely divisible random vector with non-negative components. Then for every $c_1, \ldots, c_n \geq 0$

$$\mathbb{P}(X_1 \leq c_1, \ldots, X_n \leq c_n) \geq \prod_{j=1}^n \mathbb{P}(X_j \leq c_j). \quad (2.1)$$

**Proof:** By the assumption,

$$\mathbb{E}e^{i(u,X)} = \exp \left\{ i(u,b) + \int_{\mathbb{R}^n} (e^{i(u,x)} - 1) \nu(dx) \right\}, \quad (2.2)$$

where $b \in \mathbb{R}^n_+$ and the Lévy measure $\nu$ is concentrated on $\mathbb{R}^n_+$. Let $X' = (X_1', \ldots, X_n')$ be a random vector with independent components such that $X'_j \overset{d}{=} X_j$ for each $j = 1, \ldots, n$. $X'$ is an infinitely divisible random vector with the characteristic function (2.2) where $\nu$ is replaced by $\nu'$. The Lévy measure $\nu'$ is of the form

$$\nu' = \sum_{j=1}^n \delta_0 \times \cdots \times \delta_0 \times \nu_j \times \delta_0 \times \cdots \times \delta_0,$$

where $\nu_j(A) = \nu\{x \in \mathbb{R}^n : x_j \in A \setminus \{0\}\}$. Now (2.4) can be written as

$$\mathbb{P}(X \leq c) \geq \mathbb{P}(X' \leq c), \quad c \in \mathbb{R}^n, \quad (2.3)$$

where the inequality between vectors is understood component-wise. The relation (2.2) is known as the left Slepian inequality ($X'$ dominates $X$), see [25]. By a result of Samorodnitsky and Taqqu, [25, Theorem 2.1], (2.2) holds when

$$\nu\{x \in \mathbb{R}^n : x \leq c\} \geq \nu'\{x \in \mathbb{R}^n : x \leq c\} \quad \text{for all } c \in \mathbb{R}^n \setminus \mathbb{R}^n_+ \quad (2.4)$$

and

$$\nu\{x \in \mathbb{R}^n : x \not\leq c\} \leq \nu'\{x \in \mathbb{R}^n : x \not\leq c\} \quad \text{for all } c \in \mathbb{R}^n_+. \quad (2.5)$$
Condition (2.2) holds trivially since $\nu$ and $\nu'$ are concentrated on $\mathbb{R}_+^n$. To verify (2.2), observe that for every $c \in \mathbb{R}_+^n$

$$
\nu\{x \in \mathbb{R}^n : x \not\leq c\} = \nu\left(\bigcup_{j=1}^n \{x \in \mathbb{R}^n : x_j > c_j\}\right) \leq \sum_{j=1}^n \nu\{x \in \mathbb{R}^n : x_j > c_j\}
$$

$$
= \sum_{j=1}^n \nu'((\mathbb{R} \times \cdots \times \mathbb{R} \times (c_j, \infty)) \times \mathbb{R} \times \cdots \times \mathbb{R})
$$

$$
= \nu'\{x \in \mathbb{R}^n : x \not\leq c\}.
$$

This proves (2.2) and so it does (2.4). □

2.3 Bounds by Feynman-Kac functionals

Given a bounded open domain $D \subset \mathbb{R}^3$, write

$$
\mathcal{F}_3(D) = \left\{ g \in W^{1,2}(D) : \int_D g^2(x)dx = 1 \right\}.
$$

Given a measurable function $\zeta(x)$ on $D$, we introduce the notation

$$
\lambda_\zeta(D) = \sup_{g \in \mathcal{F}_3(D)} \left\{ \int_D \zeta(x)g^2(x)dx - \frac{1}{2} \int_D |\nabla g(x)|^2dx \right\}.
$$

Clearly, $\lambda_\zeta(D) \leq \lambda_\eta(D)$ and $\lambda_\zeta(D) \leq \lambda_\zeta(D')$ whenever $\zeta(x) \leq \eta(x)$ ($x \in D$) and $D \subset D'$.

Write

$$
\tau_D = \inf\{s \geq 0 : B_s \not\in D\}.
$$

For any $r > 0$, define $T_r = \tau_D$ with $D = \{x \in \mathbb{R}^3 : |x| < r\}$.

**Lemma 2.5** For $R > 0$ and the measurable function $\zeta(x)$ defined on $B(0, R)$,

$$
\mathbb{E}_0\left[ \exp\left\{ \int_0^T \zeta(B_s)ds ; T_R \geq t \right\} \right] \geq \omega_d(R)^{-1} \int_{|x| \leq R} \mathbb{E}_x\left[ \exp\left\{ \int_0^T \zeta(B_s)ds ; T_R \geq t \right\} \right] dx
$$

where $\omega_d(R)$ is the volume of a $d$-dimensional ball of the radius $R$.

**Proof:** By a standard procedure of approximation, we may assume that $\zeta(x)$ is bounded from below on $B(0, R)$. Further, we may only consider the case when $\zeta(x) \geq 0$ on $B(0, R)$, for otherwise we consider

$$
\tilde{\zeta}(x) = \zeta(x) - \inf_{y \in B(0,R)} \zeta(y)
$$
instead.

For any $0 < r < R$, let $0 < t_r < t$ be specified later and write

$$T'_R = \inf \{ s \geq t_r; \ |B_s| \geq R \}.$$

By Markov property,

$$\mathbb{E}_0 \left[ \exp \left\{ \int_0^t \zeta(B_s)ds \right\}; \ T_R \geq t \right]$$

$$\geq \mathbb{E}_0 \left[ \exp \left\{ \int_0^{t_r} \zeta(B_s)ds \right\}; \ T_r \geq t_r, \ T'_R \geq t \right]$$

$$= \mathbb{E}_0 \left\{ \mathbf{1}_{\{T_r \geq t_r\}} \right\} \mathbb{E}_{B_{t_r}} \left[ \exp \left\{ \int_0^{t-t_r} \zeta(B_s)ds \right\}; \ T_R \geq t - t_r \right] dx$$

Recall the classic facts that $T_r$ and $B_{T_r}$ are independent and that $B_{T_r}$ is uniformly distributed on the sphere $\{|x| = r\}$. The right hand side is equal to

$$\mathbb{P}\{T_r \geq t_r\} \frac{1}{|S_r|} \int_{\{|x|=r\}} \mathbb{E}_x \left[ \exp \left\{ \int_0^{t-t_r} \zeta(B_s)ds \right\}; \ T_R \geq t - t_r \right] dx$$

where $|S_r|$ is the area of the sphere $S_r = \{x \in \mathbb{R}^d; \ |x| = r\}$.

Summarizing our argument,

$$|S_r| \left( \mathbb{P}\{T_r \geq t_r\} \right)^{-1} \mathbb{E}_0 \left[ \exp \left\{ \int_0^t \zeta(B_s)ds \right\}; \ T_R \geq t \right]$$

$$\geq \int_{\{|x|=r\}} \mathbb{E}_x \left[ \exp \left\{ \int_0^{t-t_r} \zeta(B_s)ds \right\}; \ T_R \geq t - t_r \right] dx$$

We now specifies $t_r$ as follows: Let $0 < \delta < 1$ be fixed for a moment. Define $t_r = r^2 \delta$ for $r \leq 1$ and $t_r = \delta$ for $r > 1$. Integrating the variable $r$ over $[0, R]$ on the both sides,

$$\left( \int_0^R |S_r| \left( \mathbb{P}\{T_r \geq t_r\} \right)^{-1} dr \right) \mathbb{E}_0 \left[ \exp \left\{ \int_0^t \zeta(B_s)ds \right\}; \ T_R \geq t \right]$$

$$\geq \int_{\{|x| \leq R\}} \mathbb{E}_x \left[ \exp \left\{ \int_0^{t-\delta} \zeta(B_s)ds \right\}; \ T_R \geq t \right] dx$$

By the classic bound

$$\mathbb{P}\{T_r \geq t_r\} \geq \exp \left\{ - \frac{C t_r}{r^2} \right\}$$
we have
\[
\lim_{\delta \to 0^+} \int_0^R |S_r| \left( \mathbb{P}\{T_r \geq t_r\} \right)^{-1} dr = \int_0^R |S_r| dr = \omega_d(R)
\]
Letting \( \delta \to 0^+ \) on the both sides of (2.3) completes the proof. \( \square \)

**Lemma 2.6** Let \( \zeta(x) \) be defined on the open domain \( D \subset \mathbb{R}^d \) with \( K = \sup_{x \in D} \zeta(x) < \infty \).
\[
\int_D dx \mathbb{E}_x \left[ \exp \left\{ \int_0^t \zeta(B_s) ds \right\}; \tau_D \geq t \right] \geq (2\pi)^{d/2} \exp \left\{ (t+1)\lambda_\zeta(D) - K \right\}
\]

**Proof:** By a standard procedure of approximation we may assume that \( \zeta(\cdot) \) is Hölder continuous. By Feynman-Kac representation
\[
u(t,x) = \mathbb{E}_x \left[ \exp \left\{ \int_0^t \zeta(B_s) ds \right\}; \tau_D \geq t \right]
\]
solves the initial-boundary value problem
\[
\begin{align*}
\partial_t \nu(t,x) &= \frac{1}{2} \Delta \nu(t,x) + \zeta(x) \nu(t,x) \quad (t,x) \in (0,t) \times D, \\
u(0,x) &= 1 \quad x \in D, \\
u(t,x) &= 0 \quad (t,x) \in (0,\infty) \times \partial D.
\end{align*}
\]
Let \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \) be the eigenvalues of the operator \( (1/2)\Delta + \zeta \) in \( L^2(D) \) with zero boundary condition and initial value 1 in \( D \) and let \( e_k \in L^2(D) \) be an orthogonal basis corresponding to \( \{\lambda_k\} \). By (2.31) in [14] (with \( t \) being replaced by \( t+1 \)),
\[
\mathbb{E}_x \exp \left\{ \int_0^{t+1} \zeta(B_s) ds \right\} \delta_x(B_t); \tau_D \geq t \right] = \sum_{k=1}^\infty e^{t\lambda_k} e_k^2(x) \geq \exp \left\{ (t+1)\lambda_1 e_1^2(x) \right\}.
\]
Noticing the fact that \( \lambda_1 = \lambda_\zeta(D) \) and integrating both sides we have
\[
\int_D \mathbb{E}_x \exp \left\{ \int_0^{t+1} \zeta(B_s) ds \right\} \delta_x(B_t); \tau_D \geq t \right] dx \geq \exp \left\{ (t+1)\lambda_\zeta(D) \right\}.
\]
On the other hand,
\[
\mathbb{E}_x \left[ \exp \left\{ \int_0^{t+1} \zeta(B_s) ds \right\} \delta_x(B_{t+1}); \tau_D \geq t + 1 \right] 
\leq e^K \mathbb{E}_x \left[ \exp \left\{ \int_0^t \zeta(B_s) ds \right\} \delta_x(B_{t+1}); \tau_D \geq t \right] 
= e^K \mathbb{E}_x \left[ \exp \left\{ \int_0^t \zeta(B_s) ds \right\} p_1(B_t - x); \tau_D \geq t \right]
\]
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So the conclusion follows from the bound

\[ p_1(y) = \frac{1}{(2\pi)^{d/2}} \exp \left\{ -\frac{|y|^2}{2} \right\} \leq \frac{1}{(2\pi)^{d/2}} \quad y \in \mathbb{R}^d \]

□

**Lemma 2.7** For any \( \delta > 0 \) with \( \{|x| \leq \delta\} \subset D \),

\[
\mathbb{E}_0 \left[ \exp \left\{ \theta \int_0^t \mathbb{V}(B_s) \, ds \right\}; \, \tau_D \geq 2t \right] \leq \exp \left\{ \theta t \sup_{|x| \leq \delta/2} \mathbb{V}_{\delta/1}(x) \right\} \exp \left\{ t \lambda_{2\delta}(D) \right\}
\]

conditioning on the event \( \{\omega \{ |x| \leq \delta \} = 0 \} \).

**Proof:** Notice that \( \alpha(\lambda) = 0 \) for \( \lambda \geq 3 \). Thus, on the event \( \{\omega (|x| \leq \delta) = 0 \} \),

\[
\int_{\mathbb{R}^3} \frac{\alpha(6\delta^{-1}|x|)}{|x|^2} \omega(dy) \leq \int_{\{|y| \geq \delta\}} \frac{1_{\{|y-x| \leq \delta/2\}}}{|x|^2} \omega(dy) = 0
\]

whenever \( |x| \leq \delta/2 \). Consequently,

\[ \mathbb{V}(x) = -C_{\delta} + \mathbb{V}_{\delta/1}(x) \leq \mathbb{V}_{\delta/1}(x) \]

where

\[ C_{\delta} = \int_{\mathbb{R}^3} \frac{\alpha(6\delta^{-1}|x|)}{|x|^2} dx \]

For any \( r < \delta/2 \), therefore

\[
\int_0^{T_r \wedge t} \mathbb{V}(B_s) \, ds = \int_0^{T_r \wedge t} \mathbb{V}_{\delta/1}(B_s) \, ds \leq t \sup_{|x| \leq \delta} |\mathbb{V}_{\delta/6,1}(x)|
\]

Thus,

\[
\mathbb{E}_0 \left[ \exp \left\{ \theta \int_0^t \mathbb{V}(B_s) \, ds \right\}; \, \tau_D \geq 2t \right] \leq \mathbb{E}_0 \left[ \exp \left\{ \theta \int_0^t \mathbb{V}(B_s) \, ds \right\}; \, T_r \leq t, \, \tau_D \geq 2t \right] + \exp \left\{ t \sup_{|x| \leq \delta} |\mathbb{V}_{\delta/6,1}(x)| \right\}
\]
Write $\tau_D' = \inf\{t \geq T_r; B_s \notin D\}$. By Markov property,

$$
\mathbb{E}_0 \left[ \exp \left\{ \theta \int_0^t \nabla(B_s) ds \right\}; \ T_r \leq t, \ \tau_D \geq 2t \right]
$$

$$
\leq \mathbb{E}_0 \left[ \exp \left\{ \theta T_r \left( \sup_{|x| \leq \delta} |\nabla_{\delta,1}(x)| - C_\delta \right) \right\} \exp \left\{ \theta \int_{T_r}^t \nabla(B_s) ds \right\}; \ T_r \leq t, \ \tau_D' \geq 2t \right]
$$

$$
= \mathbb{E}_0 \left[ \exp \left\{ \theta T_r \left( \sup_{|x| \leq \delta} |\nabla_{\delta,1}(x)| - C_\delta \right) \right\} u_0(t - T_r, B_{T_r}); \ T_r \leq t \right]
$$

where

$$
u_0(s, x) = \mathbb{E}_x \left[ \exp \left\{ \theta \int_0^s \nabla(B_u) du \right\}; \ \tau_D \geq t + s \right]
$$

$$
\leq \mathbb{E}_x \left[ \exp \left\{ \theta \int_0^s \nabla(B_u) du \right\}; \ \tau_D \geq t \right] \equiv u_1(s, x) \quad (0 \leq s \leq t)
$$

Notice that on $\{T_r \leq t\}$

$$
u_1(t - T_r, B_{T_r}) \leq \exp \left\{ - \theta T_r \inf_{x \in D} \nabla(x) \right\} \mathbb{E}_{B_{T_r}} \left[ \exp \left\{ \theta \int_0^t \nabla(B_u) du \right\}; \ \tau_D \geq t \right]
$$

$$
= \exp \left\{ - \theta T_r \inf_{x \in D} \nabla(x) \right\} u_2(t, B_{T_r}) \quad (\text{say})
$$

Summarizing our estimate,

$$
\mathbb{E}_0 \left[ \exp \left\{ \theta \int_0^t \nabla(B_s) ds \right\}; \ T_r \leq t, \ \tau_D \geq 2t \right]
$$

$$
\leq \mathbb{E}_0 \left[ \exp \left\{ \theta T_r \left( \sup_{|x| \leq \delta} |\nabla_{\delta,1}(x)| - C_\delta - \inf_{x \in D} \nabla(x) \right) \right\} u_2(t, B_{T_r}) \right]
$$

Recall the classic facts that $T_r$ and $B_{T_r}$ are independent and that $B_{T_r}$ is uniformly distributed on the sphere $\{|x| = r\}$. So the right hand side is equal to

$$
\mathbb{E}_0 \exp \left\{ \theta T_r \left( \sup_{|x| \leq \delta} |\nabla_{\delta,1}(x)| - C_\delta - \inf_{x \in D} \nabla(x) \right) \right\} \frac{1}{4\pi r^2} \int_{|x|=r} u_2(t, x) dx
$$

Using fact that $\{|x| \leq \delta\} \subset D$ and the bound

$$
-\nabla(x) \leq C_\delta - \nabla_{\delta,1}(x) \leq C_\delta + |\nabla_{\delta,1}(x)|
$$
we have that
\[
\mathbb{E}_0 \exp \left\{ \theta T_r \left( \sup_{|x| \leq \delta} |\nabla_{\tilde{\pi},1}(x)| - C_\delta - \inf_{x \in D} \nabla V(x) \right) \right\} \leq \mathbb{E}_0 \exp \left\{ 2\theta T_r \sup_{x \in D} |\nabla_{\tilde{\pi},1}(x)| \right\}
\]
\[
\leq \mathbb{E}_0 \exp \left\{ 2\theta T_\delta \sup_{x \in D} |\nabla_{\tilde{\pi},1}(x)| \right\} = \mathbb{E}_0 \exp \left\{ \sqrt{2d} \theta T_1 \sup_{x \in D} |\nabla_{\tilde{\pi},1}(x)| \right\}
\]

Here we have used the fact that \( T_r \leq T_\delta \frac{d}{2} = \sqrt{\frac{d}{2}} T_1 \).

By (2.3), we conclude that
\[
(4\pi r^2) \mathbb{E}_0 \left[ \exp \left\{ \theta \int_0^t \nabla(B_s) ds \right\} ; \tau_D \geq 2t \right]
\leq \mathbb{E}_0 \exp \left\{ \sqrt{2d} \theta T_1 \sup_{x \in D} |\nabla_{\tilde{\pi},1}(x)| \right\} \int_{\{|x|=r\}} u_2(t, x) dx
\]
\[
+ (4\pi r^2) \exp \left\{ t \sup_{|x| \leq \delta} |\nabla_{\tilde{\pi},1}(x)| \right\}
\]

Integrating the variable \( r \) over \([0, \delta/2]\) on the both sides,
\[
\mathbb{E}_0 \left[ \exp \left\{ \theta \int_0^t \nabla(B_s) ds \right\} ; \tau_D \geq 2t \right]
\leq \frac{6}{\pi\delta^3} \mathbb{E}_0 \exp \left\{ \sqrt{2d} \theta T_1 \sup_{x \in D} |\nabla_{\tilde{\pi},1}(x)| \right\} \int_{\{|x| \leq r\}} u_2(t, x) dx
\]
\[
+ \exp \left\{ t \sup_{|x| \leq \delta} |\nabla_{\tilde{\pi},1}(x)| \right\}
\]

Finally, the desired conclusion follows from the bound
\[
\int_{\{|x| \leq r\}} u_2(t, x) dx \leq \int_D u_2(t, x) dx \leq |D| \exp \left\{ t \lambda_\theta V(D) \right\}
\]
where the second step follows from Lemma 4.1 in [6]. □

3 Lower bounds

We establish the lower bounds requested by Theorem 1.1, Theorem 1.2 and Theorem 1.3. Let \( t \) be either fixed (as in Theorem 1.1) or increase to infinity (as in Theorem 1.2 and Theorem 1.3). Let \( \epsilon \to 0 \) and \( R \to \infty \) either as sequences (when \( t \) is fixed) or as functions of \( t \) (when \( t \to \infty \)) and will be specified later according to the context.
By Brownian scaling,
\[
\mathbb{E}_0 \exp \left\{ \theta \int_0^t \nabla(B_s) ds \right\} \geq \mathbb{E} \exp \left\{ \theta \int_0^{te^{-2/3}} \nabla_\epsilon(B_s) ds \right\} \\
\geq \left( \frac{4}{3} \pi R^3 \right)^{-1} \int_{B(0,R)} \mathbb{E}_x \left[ \exp \left\{ \theta \int_0^{te^{-2/3}} \nabla_\epsilon(B_s) ds \right\}; T_R \geq te^{\epsilon^{-2/3}} \right] dx
\]
where
\[
\nabla_\epsilon(x) = \int_{\mathbb{R}^d} \frac{1}{|y-x|^2} \left[ \omega(\epsilon dy) - \epsilon dy \right]
\]
and the inequality follows from Lemma 2.5.

Let \( r > 0 \) and \( a > 0 \) be two large but fixed numbers with \( r < a \). Consider the decomposition
\[
\nabla_\epsilon(x) = \nabla_{a,\epsilon}(x) + V_{a,\epsilon}(x) - \epsilon C_a
\]
where \( \nabla_{a,\epsilon}(x) \) is defined as in section 2.1,
\[
V_{a,\epsilon}(x) = \int_{\mathbb{R}^3} \frac{\alpha(a^{-1}|y-x|)}{|y-x|^2} \omega(\epsilon dy) \quad \text{and} \quad C_a = \int_{\mathbb{R}^d} \frac{\alpha(a^{-1}|x|)}{|x|^2} dx
\]
We have
\[
\mathbb{E}_0 \exp \left\{ \theta \int_0^t \nabla(B_s) ds \right\} \geq \mathbb{E} \exp \left\{ \theta \int_0^{te^{-2/3}} \nabla_\epsilon(B_s) ds \right\} \\
\geq \left( \frac{4}{3} \pi R^3 \right)^{-1} \exp \left\{ -\theta t e^{-2/3} \left( C_a \epsilon + \sup_{x \in B(0,R)} |\nabla_{a,\epsilon}(x)| \right) \right\} \\
\times \int_{B(0,R)} \mathbb{E}_x \left[ \exp \left\{ \theta \int_0^{te^{-2/3}} V_{a,\epsilon}(B_s) ds \right\}; T_R \geq te^{\epsilon^{-2/3}} \right] dx
\]
Let \( \delta > 0 \) be a small number satisfying \( r + \delta < a \). For any \( z \in 2r\mathbb{Z}^3 \cap B(0, R - r) \),
\[
\theta V_{a,\epsilon}(B_s) \geq \theta \int_{\{|y-z|\leq \delta\}} \frac{1}{|y-B_s|^2} \omega(\epsilon dy) \geq \theta \omega(B(\epsilon^{1/3} z, \epsilon^{1/3} \delta)) \frac{1}{(|z-B_s| + \delta)^2} \equiv \zeta_\epsilon(B_s)
\]
on the event \( \{ B_s \in B(z,r) \} \). Consequently,
\[
\int_{B(0,R)} \mathbb{E}_x \left[ \exp \left\{ \theta \int_0^{te^{-2/3}} V_{a,\epsilon}(B_s) ds \right\}; T_R \geq te^{\epsilon^{-2/3}} \right] dx \\
\geq \int_{B(z,r)} \mathbb{E}_x \left[ \exp \left\{ \theta \int_0^{te^{-2/3}} V_{a,\epsilon}(B_s) ds \right\}; \tau_{B(z,r)} \geq te^{\epsilon^{-2/3}} \right] dx \\
\geq \int_{B(z,r)} \mathbb{E}_x \left[ \exp \left\{ \theta \int_0^{te^{-2/3}} \zeta_\epsilon(B_s) ds \right\}; \tau_{B(z,r)} \geq te^{\epsilon^{-2/3}} \right] dx \\
\geq \exp \left\{ -\delta^{-2} \theta \omega(B(\epsilon^{1/3} z, \epsilon^{1/3} \delta)) \right\} \exp \left\{ te^{\epsilon^{-2/3}} \lambda_{\zeta_\epsilon}(B(z,r)) \right\}
\]
where the last step follows from Lemma 2.6 with the observation
\[
\sup_x \xi^z(x) = \delta^{-2} \theta \omega \left( B(\varepsilon^{1/3} z, \varepsilon^{1/3} \delta) \right)
\]

By substitution \(g(x) \mapsto g(x - z)\),
\[
\lambda^z(B(z, r)) = \sup_{g \in \mathcal{F}_3(B(z, r))} \left\{ \omega \left( B(\varepsilon^{1/3} z, \varepsilon^{1/3} \delta) \right) \theta \int_{B(z, r)} \frac{g^2(x)}{|x - z| + \delta} dx - \frac{1}{2} \int_{B(z, r)} |\nabla g(x)|^2 dx \right\}
\]
\[
= \sup_{g \in \mathcal{F}_3(B(0, r))} \left\{ \omega \left( B(\varepsilon^{1/3} z, \varepsilon^{1/3} \delta) \right) \theta \int_{B(0, r)} \frac{g^2(x)}{|x| + \delta} dx - \frac{1}{2} \int_{B(0, r)} |\nabla g(x)|^2 dx \right\}
\]
\[
= Q_{r, \delta} \left( \omega \left( B(\varepsilon^{1/3} z, \varepsilon^{1/3} \delta) \right) \theta \right)
\]
where the function \(Q_{a, \delta}(\cdot)\) is defined as
\[
Q_{r, \delta}(\theta) = \sup_{g \in \mathcal{F}_3(B(0, r))} \left\{ \theta \int_{B(0, r)} \frac{g^2(x)}{|x| + \delta} dx - \frac{1}{2} \int_{B(0, r)} |\nabla g(x)|^2 dx \right\}
\]

Summarizing our estimates since (3),
\[
\mathbb{E}_0 \exp \left\{ \theta \int_0^t \bar{V}(B_s) ds \right\} \geq \left( \frac{4}{3} \pi R^3 \right)^{-1} \exp \left\{ - \theta t \varepsilon^{-2/3} \left( C_a \varepsilon + \sup_{x \in B(0, R)} |\bar{V}_{a, \varepsilon}(x)| \right) \right\} \times \exp \left\{ - \delta^{-2} \theta \omega \left( B(\varepsilon^{1/3} z, \varepsilon^{1/3} \delta) \right) \right\} \exp \left\{ t \varepsilon^{-2/3} Q_{r, \delta} \left( \omega \left( B(\varepsilon^{1/3} z, \varepsilon^{1/3} \delta) \right) \theta \right) \right\}
\]

Taking maximum over \(z \in 2r\mathbb{Z}^3 \cap B(0, R - r)\) on the right hand sides,
\[
\mathbb{E}_0 \exp \left\{ \theta \int_0^t \bar{V}(B_s) ds \right\} \geq \left( \frac{4}{3} \pi R^3 \right)^{-1} \exp \left\{ - \theta t \varepsilon^{-2/3} \left( C_a \varepsilon + \sup_{x \in B(0, R)} |\bar{V}_{a, \varepsilon}(x)| \right) \right\} \times \exp \left\{ - \delta^{-2} \theta \max_{z \in 2r\mathbb{Z}^3 \cap B(0, R - r)} \omega \left( B(\varepsilon^{1/3} z, \varepsilon^{1/3} \delta) \right) \right\} \times \exp \left\{ t \varepsilon^{-2/3} Q_{r, \delta} \left( \max_{z \in 2r\mathbb{Z}^3 \cap B(0, R - r)} \omega \left( B(\varepsilon^{1/3} z, \varepsilon^{1/3} \delta) \right) \right) \right\}
\]

### 3.1 Lower bound for Theorem 1.1

We show that when \(\theta > 1/16\),
\[
\mathbb{E}_0 \exp \left\{ \theta \int_0^t \bar{V}(B_s) ds \right\} = \infty \text{ a.s. } \forall t > 0
\]
Let \( t \) be fixed. Taking \( \epsilon = 2^{-3n} \) and \( R = 2^{2n} \) in (3.1) gives
\[
\mathbb{E}_0 \exp \left\{ \theta \int_0^t V(B_s) ds \right\} \\
\geq \left( \frac{4}{3} \pi 2^{6n} \right)^{-1} \exp \left\{ - \theta t 2^{2n} \left( C_a 2^{-3n} + \sup_{x \in B(0,2^{2n})} |V_{a,2^{-3n}}(x)| \right) \right\} \\
\times \exp \left\{ - \delta^{-2} \theta \max_{z \in 2r\mathbb{Z}^3 \cap B(0,2^{2n} - r)} \omega(B(2^{-n}z, 2^{-n}\delta)) \right\} \\
\times \exp \left\{ t 2^{2n} Q_{r,\delta} \left( \theta \max_{z \in 2r\mathbb{Z}^3 \cap B(0,2^{2n} - r)} \omega(B(2^{-n}z, 2^{-n}\delta)) \right) \right\}
\]

By (2.2,
\[
\lim_{n \to \infty} \sup_{x \in B(0,2^{2n})} |V_{a,2^{-3n}}(x)| = 0 \quad \text{a.s.}
\]
when \( a > 0 \) is sufficiently large.

We now prove that
\[
\lim_{n \to \infty} \sup_{z \in 2r\mathbb{Z}^3 \cap B(0,2^{2n} - r)} \omega(B(2^{-n}z, 2^{-n}\delta)) = 2 \quad \text{a.s.}
\]

By homogeneity and increment independence of the Poisson field, The random variables
\[
\omega(B(2^{-n}z, 2^{-n}\delta)); \quad z \in 2r\mathbb{Z}^3 \cap B(0,2^{2n} - r)
\]
are i.i.d’s. Hence,
\[
\mathbb{P}\left\{ \max_{z \in 2r\mathbb{Z}^3 \cap B(0,2^{2n} - r)} \omega(B(2^{-n}z, 2^{-n}\delta)) \geq 3 \right\} \\
\leq \# \left\{ 2r\mathbb{Z}^3 \cap B(0,2^{2n} - r) \right\} \mathbb{P}\left\{ \omega(B(0,2^{-n}\delta)) \geq 3 \right\} \\
\leq C 2^{6n} \left( 2^{-n}\delta \right)^3 = O \left( 2^{-3n} \right)
\]
Thus,
\[
\sum_n \mathbb{P}\left\{ \max_{z \in 2r\mathbb{Z}^3 \cap B(0,2^{2n} - r)} \omega(B(2^{-n}z, 2^{-n}\delta)) \geq 3 \right\} < \infty
\]

By Borel-Cantelli lemma and by the fact that the random variable \( \max_{z \in 2r\mathbb{Z}^3 \cap B(0,2^{2n} - r)} \omega(B(z, 2^{-n}\delta)) \) takes integer-values,
\[
\lim_{n \to \infty} \sup_{z \in 2r\mathbb{Z}^3 \cap B(0,2^{2n} - r)} \omega(B(2^{-n}z, 2^{-n}\delta)) \leq 2 \quad \text{a.s.}
\]
On the other hand, write $A_n = B(0, 2^{2n} - r) \setminus B(0, 2^{2(n-1)})$.

$$
P\left\{ \max_{z \in 2r\mathbb{Z}^3 \cap A_n} \omega(B(2^{-n}z, 2^{-n}\delta)) \leq 1 \right\} \\
= \left( 1 - P\left\{ \omega(B(0, 2^{-n}\delta)) \geq 2 \right\} \right)^{\#\{2r\mathbb{Z}^3 \cap A_n\}} \\
\leq \left( 1 - c\delta^32^{-6n} \right)^{\#\{2r\mathbb{Z}^3 \cap A_n\}} \leq \exp\{-c_0\delta^3\}
$$

So we have that

$$
\sum_n P\left\{ \max_{z \in 2r\mathbb{Z}^3 \cap A_n} \omega(B(2^{-n}z, 2^{-n}\delta)) \geq 2 \right\} = \infty
$$

Notice that the sequence

$$
\max_{z \in 2r\mathbb{Z}^3 \cap A_n} \omega(B(2^{-n}z, 2^{-n}\delta)) \quad n = 1, 2, \cdots
$$

is an independent sequence. By Borel-Cantelli lemma

$$
\limsup_{n \to \infty} \max_{z \in 2r\mathbb{Z}^3 \cap B(0, 2^{2n} - r)} \omega(B(2^{-n}z, 2^{-n}\delta)) \geq \limsup_{n \to \infty} \max_{z \in 2r\mathbb{Z}^3 \cap A_n} \omega(B(2^{-n}z, 2^{-n}\delta)) \geq 2 \quad \text{a.s.}
$$

By the fact that $\theta > 16^{-1}$ and by Lemma 5.1,

$$
\lim_{\delta \to 0} Q_{a,\delta}(2\theta) = \sup_{g \in F_3} \left\{ 2\theta \int_{\mathbb{R}^3} \frac{g^2(x)}{|x|^2} dx - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla g(x)|^2 dx \right\} = \infty
$$

Therefore, one can take $\delta$ sufficiently small, and $a$ sufficiently large, so we have $Q_{a,\delta}(2\theta) \geq 1$. Finally, the requested (3.1) follows from (3.1), (3.1), (3.1). □

### 3.2 Lower bound for Theorem 1.2

Recall that $0 < \theta < 1/16$ and $k = [(8\theta)^{-1}]$. We prove

$$
\limsup_{t \to \infty} t^{-\frac{k+1}{k-1}} l(t)^{-\frac{2}{(k-1)}} \log \mathbb{E}_0 \exp \left\{ \theta \int_0^t V(B_s) ds \right\} = \infty \quad \text{a.s.}
$$

under the assumption

$$
\int_1^\infty \frac{dt}{t \cdot l(t)} = \infty
$$
Let $u > 0$ be fixed but arbitrary. Taking $t_n = 2^n$, $\epsilon = \epsilon_n = u^3(t_n^3 l(t_n))^{-\frac{1}{k+1}}$, $R = R_n = \frac{k+1}{t_n} l(t_n)^{\frac{2}{3(k+1)}}$ in (3.1) gives

$$
\mathbb{E}_0 \exp \left\{ \theta \int_0^{t_n} \nabla(B_s) ds \right\} 
\geq \left( \frac{4}{3} \pi R_n \right)^{-1} \exp \left\{ -\theta u^{2^{k+1}} l(t_n)^{\frac{2}{3(k+1)}} \left( C_d \epsilon_n + \sup_{x \in B(0, R_n)} |\nabla_{a, \epsilon_n}(x)| \right) \right\} 
\times \exp \left\{ -\delta^{-2} \theta \max_{z \in 2r\mathbb{Z}^3 \cap B(0, R_n - r)} \omega(B(\epsilon_n^{1/3} z, \epsilon_n^{1/3} \delta)) \right\} 
\times \exp \left\{ u^{2^{k+1}} l(t_n)^{\frac{2}{3(k+1)}} Q_{r, \delta} \left( \theta \max_{z \in 2r\mathbb{Z}^3 \cap B(0, R_n - r)} \omega(B(\epsilon_n^{1/3} z, \epsilon_n^{1/3} \delta)) \right) \right\}
$$

By (2.2),

$$
\lim_{n \to \infty} \sup_{x \in B(0, R_n)} \left| \nabla_{a, \epsilon_n}(x) \right| = 0 \quad \text{a.s.}
$$
as $a > 0$ is sufficiently large.

In addition,

$$
\mathbb{P} \left\{ \max_{z \in 2r\mathbb{Z}^3 \cap B(0, R_n - r)} \omega(B(\epsilon_n^{1/3} z, \epsilon_n^{1/3} \delta)) \geq k + 2 \right\} 
\leq C t_n^{\frac{3}{k+1}} l(t_n)^{-\frac{k}{k+1}} \mathbb{P} \left\{ \omega(B(0, \epsilon_n^{1/3} \delta)) \geq k + 2 \right\} 
\leq C t_n^{\frac{3}{k+1}} l(t_n)^{-\frac{k}{k+1}}
$$

Consequently,

$$
\sum_n \mathbb{P} \left\{ \max_{z \in 2r\mathbb{Z}^3 \cap B(0, R_n - r)} \omega(B(\epsilon_n^{1/3} z, \epsilon_n^{1/3} \delta)) \geq k + 2 \right\} < \infty.
$$

By Borel-Cantelli lemma,

$$
\limsup_{n \to \infty} \max_{z \in 2r\mathbb{Z}^3 \cap B(0, R_n - r)} \omega(B(\epsilon_n^{1/3} z, \epsilon_n^{1/3} \delta)) \leq k + 1 \quad \text{a.s.}
$$

On the other hand, let $A_n = B(0, R_n - r) \setminus B(0, R_{n-1} - r)$.

$$
\mathbb{P} \left\{ \max_{z \in 2r\mathbb{Z}^3 \cap A_n} \omega(B(\epsilon_n^{1/3} z, \epsilon_n^{1/3} \delta)) \leq k \right\} 
= \left( 1 - \mathbb{P} \left\{ \omega(B(0, \epsilon_n^{1/3} \delta)) \geq k + 1 \right\} \right)^{#(2r\mathbb{Z}^3 \cap A_n)}
$$

Hence,

$$
\mathbb{P} \left\{ \max_{z \in 2r\mathbb{Z}^3 \cap A_n} \omega(B(\epsilon_n^{1/3} z, \epsilon_n^{1/3} \delta)) \geq k + 1 \right\} 
\sim #(2r\mathbb{Z}^3 \cap A_n) \mathbb{P} \left\{ \omega(B(0, \epsilon_n^{1/3} \delta)) \geq k + 1 \right\} \geq c_0 l(t_n)^{-1}
$$

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where \( c_0 > 0 \) is a constant independent of \( n \). By (3.2),
\[
\sum_n l(t_n)^{-1} = \infty
\]

By Borel-Cantelli lemma,
\[
\limsup_{n \to \infty} \max_{z \in 2rZ^3 \cap B(0, R_n - r)} \omega(B(\epsilon_n^{1/3} z, \epsilon_n^{1/3} \delta)) \geq \limsup_{n \to \infty} \max_{z \in 2rZ^3 \cap A_n} \omega(B(\epsilon_n^{1/3} z, \epsilon_n^{1/3} \delta)) \geq k + 1 \quad a.s.
\]

Notice that \( (k+1)\theta > 1/8 \). By Lemma 5.1, we can \( \delta \) sufficiently small and \( r \) sufficiently large so \( Q_{r,\delta}((k+1)\theta) \geq 1 \). By (3.2), (3.2), (3.2), (3.2),
\[
\limsup_{n \to \infty} t_n^{k+1} l(t_n)^{-3/4} \log \mathbb{E}_0 \exp \left\{ \theta \int_0^{t_n} \bar{V}(B_s) ds \right\} \geq u^2 \quad a.s.
\]

Since \( u > 0 \) can be arbitrarily large, we have proved (3.2). \( \square \)

### 3.3 Lower bound for Theorem 1.3

We prove that
\[
\liminf_{t \to \infty} t^{-3/4} l(t)^{-3/4} \log \mathbb{E}_0 \exp \left\{ \theta \int_0^{t} \bar{V}(B_s) ds \right\} = \infty \quad a.s.
\]

under the assumption that
\[
\int_1^\infty \frac{1}{t} \exp \left\{ - c \cdot l(t) \right\} dt < \infty \quad \forall c > 0
\]

Taking \( t_n = 2^n \), \( \epsilon = \epsilon_n = u^3 (t_n^{-3} l(t_n))^{1/4} \), \( R = R_n = t_n^{k+1} l(t_n)^{-3/4} \) in (3.1) gives
\[
\mathbb{E}_0 \exp \left\{ \theta \int_0^{t_n} \bar{V}(B_s) ds \right\} \geq \left( \frac{4}{3} \pi R_n \right)^{-1} \exp \left\{ - \theta u^2 t_n^{k+1} l(t_n)^{-3/4} \left( C_a \epsilon_n + \sup_{x \in B(0, R_n)} |\bar{V}_{a,\epsilon_n}(x)| \right) \right\} \times \exp \left\{ - \delta^{-2} \theta \max_{z \in 2rZ^3 \cap B(0, R_n-r)} \omega(B(\epsilon_n^{1/3} z, \epsilon_n^{1/3} \delta)) \right\} \times \exp \left\{ u^2 t_n^{k+1} l(t_n)^{-3/4} Q_{r,\delta} \left( \delta \max_{z \in 2rZ^3 \cap B(0, R_n-r)} \omega(B(\epsilon_n^{1/3} z, \epsilon_n^{1/3} \delta)) \right) \right\}
\]

Notice that (3.2) and (3.2) remain true in this setting, and that \( Q_{r,\delta}((k+1)\theta) \geq 1 \) for small \( \delta > 0 \) and large \( r \). All we need is to show that for any \( \delta > 0 \),
\[
\liminf_{n \to \infty} \max_{z \in 2rZ^3 \cap B(0, R_n-r)} \omega(B(\epsilon_n^{1/3} z, \epsilon_n^{1/3} \delta)) \geq k + 1 \quad a.s.
\]
Indeed, by independence
\[
\mathbb{P}\left\{ \max_{z \in 2rZ \cap B(0,R_n-r)} \omega(B(e_n^{1/3}z, e_n^{1/3} \delta)) \leq k \right\} \\
= \left(1 - \mathbb{P}\left\{ \omega(B(0, e_n^{1/3} \delta)) \geq k + 1 \right\} \right)^{\#\{2rZ \cap B(0,R_n-r)\}}
\]
By the fact that
\[
\mathbb{P}\left\{ \omega(B(0, e_n^{1/3} \delta)) \geq k + 1 \right\} \sim \frac{1}{(k+1)!} \left(\frac{4}{3} \pi \delta^3 e_n\right)^{k+1}
\]
there is a constant \(c(k, \delta) > 0\) such that
\[
\mathbb{P}\left\{ \max_{z \in 2rZ \cap B(0,R_n-r)} \omega(B(e_n^{1/3}z, e_n^{1/3} \delta)) \leq k \right\} \leq \exp\left\{ -c(k, \delta)l(t_n) \right\}
\]
for large \(n\). By (3.3),
\[
\sum_n \exp\left\{ -c(k, \delta)l(t_n) \right\} < \infty
\]
Hence, (3.3) follows from Borel-Cantelli lemma. \(\Box\)

4 Upper bounds

In this section we install the upper bounds requested by Theorem 1.1, Theorem 1.2, and Theorem 1.3. Through this section \(0 < \theta < 1/16\) and \(l(t)\) is a non-decreasing, slow-varying such that \(l(t) \uparrow \infty\). Recall that \(k = [(8\theta)^{-1}]\). For each \(R > 0\), write \(Q_R = (-R, R)^d\).

4.1 Asymptotics for the principal eigenvalues

By \(0 < \theta < 1/16\) we have that \(k \geq 2\). Write
\[
R_k(t) = \begin{cases} 
    t^{\frac{k}{\pi-2}} l(t)^{\frac{2}{\pi-2j}} & \text{when } k \geq 3 \\
    t^3 l(t)^{2/3} & \text{when } k = 2.
\end{cases}
\]

Lemma 4.1
\[
\lim_{t \to \infty} t^{-\frac{2}{\pi-2}} l(t)^{-\frac{2}{\pi-2j}} \lambda_{\theta V}(Q_{R_k(t)}) = 0 \quad \text{a.s.}
\]
under the assumption
\[
\int_1^\infty \frac{dt}{t \cdot l(t)} < \infty
\]
Proof: We first consider the case \( k \geq 3 \). Let \( u > 0 \) and \( M > 0 \) be fixed but arbitrary. Write
\[
r(t) = M(tl(t)^{1/3})^{\frac{1}{(k-1)(k-2)}}, \quad \epsilon(t) = u^{-3}(t^{3}l(t))^{-\frac{k}{(k-1)(k-2)}}
\]
\[
\delta(t) = \epsilon(t)^{1/3}r(t) = \left(\frac{M}{u}\right)(tl(t)^{1/3})^{-\frac{1}{k-2}}
\]
Decompose \( \nabla \) as follows:
\[
\nabla(x) = \int_{\mathbb{R}^{3}} \frac{\alpha(\delta(t)-1|y-x|)}{|y-x|^{2}}[\omega(dy) - dy] + \int_{\mathbb{R}^{3}} \frac{1 - \alpha(\delta(t)-1|y-x|)}{|y-x|^{2}}[\omega(dy) - dy]
\]
For the first term
\[
\int_{\mathbb{R}^{3}} \frac{\alpha(\delta(t)-1|y-x|)}{|y-x|^{2}}[\omega(dy) - dy] \leq \int_{\mathbb{R}^{3}} \frac{\alpha(\delta(t)-1|y-x|)}{|y-x|^{2}} \omega(dy)
\]
\[
= \epsilon(t)^{-2/3} \int_{\mathbb{R}^{3}} \frac{\alpha(r(t)-1|y-x|)}{|y-x|^{2}} \omega(\epsilon(t)dy) = \epsilon(t)^{-2/3} \xi_{r,\epsilon}(\epsilon(t)-1/3;x)
\]
where
\[
\xi_{r,\epsilon}(x) = \xi_{r(t),\epsilon(t)}(x) = \int_{\mathbb{R}^{3}} \frac{\alpha(r(t)-1|y-x|)}{|y-x|^{2}} \omega(\epsilon(t)dy)
\]
As for the second term
\[
\int_{\mathbb{R}^{3}} \frac{1 - \alpha(\delta(t)-1|y-x|)}{|y-x|^{2}}[\omega(dy) - dy]
\]
\[
= a^{2}\delta(t)^{-2} \int_{\mathbb{R}^{3}} \frac{1 - \alpha(a^{-1}|y-a\delta(t)-1|x|)}{|y-a\delta(t)-1|x|^{2}}[\omega(a^{-3}\delta(t)^{3}dy) - a^{-3}\delta(t)^{3}dy]
\]
\[
= a^{2}\delta(t)^{-2} \nabla_{a,\delta(t)}(a\delta(t)-1x)
\]
where \( \delta(t) = a^{-3}\delta(t)^{3} \), the random field \( \nabla_{a,\epsilon}() \) is defined in section 2.1 and the constant \( a > 0 \) will be specified later.

By triangle inequality and by the substitution \( g(x) \mapsto \epsilon(t)^{-1/2}g(x\epsilon(t)^{-1/3}) \),
\[
\lambda_{\epsilon\nabla}(Q_{R_{k}(t)}) \leq \epsilon(t)^{-2/3}\lambda_{r_{\xi},\epsilon}(Q_{\epsilon^{-1/3}(t)R_{k}(t)}) + \theta a^{2}\delta^{-2}(t) \sup_{x \in a\delta(t)^{-1}Q_{R_{k}(t)}}|\nabla_{a,\delta(t)}(x)|
\]
By Proposition 1 in [13], there is a non-negative and continuous function \( \Phi(x) \) on \( \mathbb{R}^{d} \) whose support is contained in the 1-neighborhood of the grid \( 2r(t)\mathbb{Z}^{d} \), such that
\[
\lambda_{r_{\xi},\epsilon}(Q_{\epsilon^{-1/3}(t)R_{k}(t)}) \leq \max_{z \in 2r(t)\mathbb{Z}^{d} \cap Q_{\epsilon^{-1/3}(t)R_{k}(t) + 2r(t)}} \lambda_{r_{\xi},\epsilon}(z + Q_{r(t)+1}) \quad y \in Q_{r(t)}
\]
where \( \Phi(y) = \Phi(x+y) \). In addition, \( \Phi(x) \) is periodic with period \( 2r(t) \):
\[
\Phi(x + 2r(t)z) = \Phi(x); \quad x \in \mathbb{R}^{d}, \ z \in \mathbb{Z}^{d}
\]
and there is a constant $K > 0$ independent of $r(t)$ and $t$ such that
\[ \int_{Q_r} \Phi(x)dx \leq \frac{K}{r(t)}. \]
By periodicity, therefore,
\[ \eta(x) \equiv \frac{1}{(2r(t))^3} \int_{Q_r} \Phi^y(x)dy = \frac{1}{(2r(t))^3} \int_{Q_r} \Phi(y)dy \leq \frac{K}{8r(t)^4}. \]
So we have
\[ \lambda_{\varepsilon,r} (Q_{e^{-1/3}(t)R_k(t)}) \leq \frac{K}{8r(t)^4} + \lambda_{\varepsilon,r} - \eta (Q_{e^{-1/3}(t)R_k(t)}) \]
\[ \leq \frac{K}{8r(t)^4} + \frac{1}{(2r(t))^3} \int_{Q_{r(t)}} \lambda_{\varepsilon,r} - \Phi^y (Q_{e^{-1/3}(t)R_k(t)})dy \]
\[ \leq \frac{K}{8r(t)^4} + \max_{z \in 2r(t)Z^d \cap Q_{2e^{-1/3}(t)R_k(t)+2r(t)}} \lambda_{\varepsilon,r} (z + Q_{r(t)+1}) \]
where the second step follows from Jensen inequality.
Summarizing the estimate since (4.1),
\[ t^{-\frac{2}{3}}l(t)^{-\frac{2}{3}(k-1)} \lambda_{\theta t} (Q_{R_k(t)}) \]
\[ \leq \theta a^2 \left( \frac{u}{M} \right)^2 \sup_{x \in \delta(t)^{-1} Q_{R_k(t)}} |\nabla_{\theta \lambda(t)}(x)| + \frac{Ku^2}{8M^4} \]
\[ + u^2 (tL(t)^{1/3}) \left( \frac{2}{(k-2)^{1/2}} \right) \max_{z \in 2r(t)Z^d \cap Q_{2e^{-1/3}(t)R_k(t)+2r(t)}} \lambda_{\theta \lambda(t)} (z + Q_{r(t)+1}) \]
Take $t_n = 2^n$. By (2.2), leads to
\[ \lim_{n \to \infty} \sup_{x \in \delta(t_n)^{-1} Q_{R_k(t_n)}} |\nabla_{\theta \lambda(t)}(x)| = 0 \quad a.s. \]
when $a$ is sufficiently large.
We now prove that
\[ \mathbb{P}\left\{ \max_{z \in 2r(t_n)Z^d \cap Q_{2e^{-1/3}(t_n)R_k(t_n)+2r(t_n)}} \lambda_{\varepsilon_r(t_n),x(t_n)} (z + Q_{r(t_n)+1}) = 0 \text{ eventually in } n \right\} = 1 \]
Notice that
\[ \mathbb{P}\left\{ \max_{z \in 2r(t_n)Z^d \cap Q_{2e^{-1/3}(t_n)R_k(t_n)+2r(t_n)}} \lambda_{\theta t_r(t_n),x(t_n)} (z + Q_{r(t_n)+1}) \neq 0 \right\} \]
\[ \leq \# \left\{ 2r(t_n)Z^d \cap Q_{2e^{-1/3}(t_n)R_k(t_n)+2r(t_n)} \right\} \mathbb{P}\left\{ \lambda_{\theta t_r(t_n),x(t_n)} (Q_{r(t_n)+1}) \neq 0 \right\} \]
The truncation function $\alpha(\cdot)$ is supported on $[0, 3]$. For any $g \in \mathcal{F}_3(Q_{r(t_n)+1})$, 
\[
\int_{Q_{r(t_n)+1}} \xi_{r(t_n),\epsilon(t_n)}(x) g^2(x) dx = \int_{\mathbb{R}^3} \left[ \int_{Q_{r(t_n)+1}} \frac{\alpha(t_n)^{-1}|y - x|}{|y - x|^2} g^2(y) dy \right] \omega(\epsilon(t_n) dx) \\
= \int_{Q_{r(t_n)}} \left[ \int_{Q_{r(t_n)+1}} \frac{\alpha(a^{-1}|y - x|)}{|y - x|^2} g^2(y) dy \right] \omega(\epsilon(t_n) dx) \\
\leq \omega(\epsilon(t_n)^{1/3} Q_{5r(t_n)}) \sup_{x \in \mathbb{R}^3} \int_{Q_{r(t_n)+1}} \frac{g^2(y)}{|y - x|^2} dy
\]
when $r(t_n) \geq 1$. Therefore,
\[
\lambda_{\theta \xi_{r(t_n),\epsilon(t_n)}}(Q_{r(t_n)+1}) \\
\leq \sup_{g \in \mathcal{F}_3(Q_{r(t_n)+1})} \left\{ \omega(\epsilon(t_n)^{1/3} Q_{5r(t_n)}) \theta \sup_{x \in \mathbb{R}^3} \int_{Q_{r(t_n)+1}} \frac{\theta g^2(y)}{|y - x|^2} dy - \frac{1}{2} \int_{Q_{r(t_n)+1}} |\nabla g(y)|^2 dy \right\} \\
\leq \sup_{g \in \mathcal{F}_3} \left\{ \omega(\epsilon(t_n)^{1/3} Q_{5r(t_n)}) \theta \sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\theta g^2(y)}{|y - x|^2} dy - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla g(y)|^2 dy \right\} \\
= \sup_{x \in \mathbb{R}^3} \sup_{g \in \mathcal{F}_3} \left\{ \omega(\epsilon(t_n)^{1/3} Q_{5r(t_n)}) \theta \theta \int_{\mathbb{R}^3} \frac{g^2(y)}{|y|^2} dy - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla g(y)|^2 dy \right\} \\
= \sup_{g \in \mathcal{F}_3} \left\{ \omega(\epsilon(t_n)^{1/3} Q_{5r(t_n)}) \theta \int_{\mathbb{R}^3} \frac{g^2(y)}{|y|^2} dy - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla g(y)|^2 dy \right\}
\]
where the last step follows from shifting invariance.

Notice that $k\theta \leq 8^{-1}$. By Lemma 5.1 we obtain the bound
\[
\mathbb{P} \left\{ \max_{z \in 2r(t_n)z^d \cap Q_{2a^{-1/3}(t_n)R_{\epsilon}(t_n)+2r(t_n)}} \lambda_{\theta \xi_{a,\epsilon(t_n)}}(z + Q_{r(t_n)+1}) \neq 0 \right\} \\
\leq C t_n^{-\frac{3}{k+2}} l(t_n)^{\frac{3}{2}} \mathbb{P} \left\{ \omega(\epsilon(t_n)^{1/3} Q_{5r(t_n)}) \geq k + 1 \right\} \leq Cl(t_n)^{-1}
\]
By (4.1),
\[
\sum_n l(t_n)^{-1} < \infty
\]
Hence, (4.1) follows from Borel-Cantelli lemma.

Since the second term in (4.1) can be arbitrarily small by making $M$ sufficiently large, by (4.1) and (4.1),
\[
\lim_{n \to \infty} t_n^{-2/\pi} l(t_n)^{-\frac{2}{\pi(k-1)}} \lambda_{gV}(Q_{R_k(t_n)}) \leq 0 \quad a.s.
\]
Notice that $\lambda_{gV}(Q_{R_k(t)})$ is non-decreasing in $t$. We have completed the proof in the case $k \geq 3$. 

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The case $k = 2$ follows from the same argument with

$$r(t) = M(tl(t)^{1/3})^{1/2}, \quad \epsilon(t) = u^{-3}(t^3l(t))^{-2}$$

□

Write

$$S_k(t) = \begin{cases} t^{k/2}l(t)^{-2/(k-2)} & \text{when } k \geq 3 \\ t^3l(t)^{-2/3} & \text{when } k = 2. \end{cases}$$

**Lemma 4.2**

$$\liminf_{t \to \infty} t^{-\frac{2}{3(k-1)}} \log \left( Q_{S_k(t)} \right) = 0 \quad \text{a.s.}$$

under the assumption that there is $c_0 > 0$ such that

$$\int_1^\infty \frac{1}{t} \exp \{ -ct \} dt \begin{cases} = \infty & \text{when } c < c_0 \\ < \infty & \text{when } c > c_0. \end{cases}$$

**Proof:** We first consider the case $k \geq 3$. Let $u > 0$ and $M > 0$ be fixed but arbitrary. Write

$$r(t) = M(tl(t)^{-1/3})^{1/(k-1)}, \quad \epsilon(t) = u^{-3}(t^3l(t))^{-1/(k-1)}$$

Similar as (4.1),

$$t^{-\frac{2}{3(k-1)}} \frac{\lambda \theta V}{\lambda_\theta V} \left( Q_{S_k(t)} \right)$$

$$\leq \omega \left( \frac{u}{M} \right) \sup_{x \in \delta(t)^{-1}Q_{S_k(t)}} \left| \nabla a,\delta(t)(x) \right| + \frac{Ku^2}{8M^4}$$

$$+ u^2(tl(t)^{-1/3})^{2/(k-1)} \max_{z \in 2r(t)\mathbb{Z}^d \cap Q_{2(-1/3)(s_k(t)+2r(t))}} \lambda \theta_{\epsilon,\delta}(z + Q_{r(t)+1})$$

where the random fields $\nabla a,\epsilon(x)$ and $\xi,\delta(x)$ are defined in the same way as they are in the proof of Lemma 4.1.

Same as in the proof of Lemma 4.1,

$$\lambda \theta_{\epsilon,\delta}(z + Q_{r(t)+1}) \leq \sup_{g \in F_3} \left\{ \omega \left( \epsilon(t)^{1/3}(z + Q_{5r(t)}) \right) \theta \int_{\mathbb{R}^3} \frac{g^2(y)}{|y|^2} dy - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla g|^2 dy \right\}$$

for each $z \in 2r(t)\mathbb{Z}^d \cap Q_{2(-1/3)(s_k(t)+2r(t))}$. Thus,

$$\max_{z \in 2r(t)\mathbb{Z}^d \cap Q_{2(-1/3)(s_k(t)+2r(t))}} \lambda \theta_{\epsilon,\delta}(z + Q_{r(t)+1})$$

$$\leq \sup_{g \in F_3} \left\{ \theta \left( \max_{z \in 2r(t)\mathbb{Z}^d \cap Q_{2(-1/3)(s_k(t)+2r(t))}} \omega \left( \epsilon(t)^{1/3}(z + Q_{5r(t)}) \right) \right) \int_{\mathbb{R}^3} \frac{g^2(y)}{|y|^2} dy - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla g|^2 dy \right\}$$

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By Lemma 5.1, therefore,
\[
\max_{z \in 2r(t) \mathbb{Z}^d \cap Q_{2r^{-1/3}(t)S_k(t)+2r(t)}} \lambda_{S_{k}}(z + Q_{r(t)+1}) = 0
\]
\[
\sup_{z \in 2r(t) \mathbb{Z}^d \cap Q_{2r^{-1/3}(t)S_k(t)+2r(t)}} \omega(t^{1/3}(z + Q_{5r(t)})) \leq k
\]

Unfortunately, the random variables
\[
\omega(t^{1/3}(z + Q_{5r(t)})); \quad z \in 2r(t) \mathbb{Z}^d \cap Q_{2r^{-1/3}(t)S_k(t)+2r(t)}
\]
are not independent as. So we apply Slepian-type domination (Lemma 2.3)
\[
\mathbb{P} \left\{ \max_{z \in 2r(t) \mathbb{Z}^d \cap Q_{2r^{-1/3}(t)S_k(t)+2r(t)}} \omega(t^{1/3}(z + Q_{5r(t)})) \leq k \right\}
\geq \left( \mathbb{P} \left\{ \omega(t^{1/3}Q_{5r(t)}) \leq k \right\} \right)^\#(2r(t) \mathbb{Z}^d \cap Q_{2r^{-1/3}(t)S_k(t)+2r(t)})
\]

It is straightforward to check that
\[
\mathbb{P} \left\{ \omega(t^{1/3}Q_{5r(t)}) \geq k + 1 \right\} \sim \frac{(10u^{-1}M)^{3(k+1)}}{(k+1)!} \left( t^3 l(t)^{-1} \right)^{-\frac{k+1}{3k-2}} \quad (t \to \infty)
\]
and that
\[
\#(2r(t) \mathbb{Z}^d \cap Q_{2r^{-1/3}(t)S_k(t)+2r(t)}) \sim \left( \frac{u}{M} \right)^3 t^{\frac{3(k+1)}{k-2}} l(t)^{-\frac{3}{k-2}} \quad (t \to \infty)
\]
Hence, there is a constant \(C_k\) independent of \(u\) and \(M\) such that
\[
\mathbb{P} \left\{ \max_{z \in 2r(t) \mathbb{Z}^d \cap Q_{2r^{-1/3}(t)S_k(t)+2r(t)}} \omega(t^{1/3}(z + Q_{5r(t)})) \leq k \right\} \geq \exp \left\{ - C_k \left( \frac{M}{u} \right)^{3k} l(t) \right\}
\]
for large \(t\). In connection to (4.1), our strategy is to make \(u^2/M^4\), \(M/u\) sufficiently small, and to make \(u\) and \(M\) sufficiently large.

Fix a constant \(\tilde{c}\) satisfying
\[
\frac{k-1}{3k} c_0 < \tilde{c} < c_0
\]
Define \(\{t_n\}\) as following:
\[
t_1 = 1, \quad t_{n+1} = t_n \exp \left\{ \tilde{c} l(t_n) \right\} \quad n = 1, 2, \cdots
\]
By (2.2),
\[
\lim_{n \to \infty} \sup_{x \in Q_{2r^{-1/3}(t_n)S_k(t_n)}} |\nabla_{\alpha\delta(t_n)}(x)| = 0 \quad a.s.
\]
for sufficiently large $a$.

We now prove that

$$\Pr\left\{ \max_{z \in 2r(t_n)Z^d \cap Q_{2r^{-1/3}(t_n)S_k(t_n)+2r(t_n)}} \omega\left(\epsilon(t_n)^{1/3}(z + Q_{5r(t_n)})\right) \leq k \text{ i.o.} \right\} = 1$$

Write

$$H_n = \max_{z \in 2r(t_n)Z^d \cap Q_{2r^{-1/3}(t_n)S_k(t_n)+2r(t_n)}} \omega\left(\epsilon(t_n)^{1/3}(z + Q_{5r(t_n)})\right)$$

$$A_n = Q_{2r^{-1/3}(t_n+1)S_k(t_n+1)+2r(t_n+1)} \setminus Q_{2r^{-1/3}(t_n)S_k(t_n)+br(t_n)}$$

$$Z_n = \max_{z \in 2r(t_n)Z^d \cap A_n} \omega\left(\epsilon(t_n+1)^{1/3}(z + Q_{5r(t_n+1)})\right)$$

$$\tilde{Z}_n = \max_{z \in 2r(t_n)Z^d \cap Q_{2r^{-1/3}(t_n)S_k(t_n)+br(t_n)}} \omega\left(\epsilon(t_n+1)^{1/3}(z + Q_{5r(t_n+1)})\right)$$

where $b > 0$ is a constant which is large enough to make sure that the random variables $Z_1, Z_2, \ldots$ are independent.

We have that $H_{n+1} = \max\{Z_n, \tilde{Z}_n\}$. Notice that

$$\Pr\{\tilde{Z}_n \geq k + 1\} \leq \#\{2r(t_n)Z^d \cap Q_{2r^{-1/3}(t_n)S_k(t_n)+4r(t_n)}\} \Pr\left\{ \omega\left(\epsilon(t_n+1)^{1/3}Q_{5r(t_n+1)}\right) \geq k + 1\right\}$$

$$\leq Ct_n^{3(k+1)} l(t_n)^{-\frac{3}{k+1}} t_{n+1}^{\frac{3(k+1)}{k+3}} l(t_{n+1})^\frac{k+1}{k+3} = Cl(t_n)^{-\frac{3}{k+1}} l(t_{n+1})^\frac{k+1}{k+3} \exp\left\{ - \frac{3\bar{c}(k+1)}{k-2} l(t_n) \right\}$$

Since $l(t)$ is slow-varying,

$$l(t_{n+1}) = l\left(t_n \exp\{\bar{c}l(t_n)\}\right) \leq l(t_n) \exp\left\{ o(l(t_n)) \right\} = \exp\left\{ o(l(t_n)) \right\}$$

for large $n$. Therefore, we obtain the bound

$$\Pr\{\tilde{Z}_n \geq k + 1\} \leq C \exp\left\{ - \frac{3k\bar{c}}{k-2} l(t_n) \right\} \quad (n \to \infty)$$

For any $c > c_0$, on the other hand,

$$\infty > \int_1^\infty \frac{1}{t} \exp\left\{ - cl(t) \right\} dt = \sum_{n=1}^\infty \int_{t_n}^{t_{n+1}} \frac{1}{t} \exp\left\{ - cl(t) \right\} dt$$

$$\geq \sum_{n=1}^\infty \frac{t_{n+1} - t_n}{t_{n+1}} \exp\left\{ - cl(t_{n+1}) \right\} \geq \delta \sum_{n=1}^\infty \exp\left\{ - cl(t_{n+1}) \right\}$$

So we have that

$$\sum_n \Pr\{\tilde{Z}_n \geq k + 1\} < \infty$$
By Borel-Cantelli lemma

\[ P\{\tilde{Z}_n \leq k \text{ eventually in } n\} = 1 \]

By (4.1),

\[ P\{Z_n \leq k\} \geq P\{H_{n+1} \leq k\} \geq \exp\left\{ -C_k \left( \frac{M}{u}\right)^{3k} l(t_{n+1}) \right\} \]

Pick \( c_1 \) satisfying \( \tilde{c} < c_1 < c_0 \) and make \( M/u \) so small that

\[ C_k \left( \frac{L}{u} \right)^{3k} < c_1 - \tilde{c} \]

We have

\[
\begin{align*}
\infty &= \int_1^\infty \frac{1}{t} \exp \left\{ -c_1 l(t) \right\} dt = \sum_{n=1}^\infty \int_{t_n}^{t_{n+1}} \frac{1}{t} \exp \left\{ -c_1 l(t) \right\} dt \\
&\leq \sum_{n=1}^\infty \frac{t_{n+1} - t_n}{t_n} \exp \left\{ -c_1 l(t_n) \right\} \leq \sum_{n=1}^\infty \exp \left\{ -(c_1 - \tilde{c}) l(t_n) \right\}
\end{align*}
\]

Consequently,

\[ \sum_n P\{Z_n \leq k\} = \infty \]

Applying Borel-Cantelli lemma to the independent sequence \( \{Z_n\} \) we have

\[ P\{Z_n \leq k \text{ i.o.}\} = 1 \]

We have proved (4.1). By (4.1) and (4.1),

\[ P\left\{ \max_{z \in 2r(t_n)Z^d \cap Q_{2e^{-1/3}(t_n)S_k(t_n)+2r(t_n)}} \lambda_{\ell_t(t_n),r(t_n)}(z+Q_{r(t_n)+1}) = 0 \text{ i.o.} \right\} = 1 \]

By (4.1), (4.1), (4.1), and by the fact that \( u^2/M^4 \) can be arbitrarily small,

\[
\liminf_{n \to \infty} t_n^{-\frac{2}{3-k-1}} l(t_n)^{\frac{2}{3-k-1}} \lambda_{\theta \Gamma}(Q_{S_k(t_n)}) \leq 0 \quad a.s.
\]

By the fact that the principal eigenvalue \( \lambda_{\theta \Gamma}(Q_R) \) increases with \( R \), we have completed the proof in the case \( k \geq 3 \).

The case \( k = 2 \) follows from the same argument with

\[ r(t) = M(tl(t)^{-1/3})^{1/2}, \quad \epsilon(t) = u^{-3}(t^3 l(t)^{-1})^{-2} \]

\[ \square \]
4.2 Upper bound for Theorem 1.1

We prove that when \( \theta < 16^{-1} \),

\[
E_0 \exp \left\{ \theta \int_0^t V(B_s) ds \right\} < \infty \quad a.s.
\]

for any \( t > 0 \). By Hölder inequality, we may assume that \( \theta > \frac{1}{24} \).

Let \( l(t) \geq 0 \) be a slow-varying function satisfying (4.1) and recall the notation \( R_2(t) = t^3 l(t)^{2/3} \). Consider the decomposition

\[
E_0 \exp \left\{ \theta \int_0^t V(B_s) ds \right\} = E_0 \left[ \exp \left\{ \theta \int_0^t V(B_s) ds \right\}; \; \tau_{Q_2(t)} \geq 2t \right]
+ \sum_{n=1}^{\infty} E_0 \left[ \exp \left\{ \theta \int_0^t V(B_s) ds \right\}; \; \tau_{Q_2(2^n t)} < 2t \leq \tau_{Q_2(2^{n+1} t)} \right]
\]

Pick \( p > 1 \) with \( p \theta < 16^{-1} \) and write \( q = p(p - 1)^{-1} \). By Hölder inequality,

\[
E_0 \left[ \exp \left\{ \theta \int_0^t V(B_s) ds \right\}; \; \tau_{Q_2(2^n t)} < 2t \leq \tau_{Q_2(2^{n+1} t)} \right] \leq \left( \mathbb{P}_0 \left\{ \tau_{Q_2(2^n t)} < 2t \right\} \right)^{1/q} \left( E_0 \left[ \exp \left\{ p \theta \int_0^t V(B_s) ds \right\}; \; \tau_{Q_2(2^{n+1} t)} \geq 2t \right] \right)^{1/p}
\]

Let \( \delta > 0 \) be a small number and condition on the event \( \{ \omega(B(0, \delta)) = 0 \} \). Applying Lemma 2.7,

\[
E_0 \exp \left\{ \theta \int_0^t V(B_s) ds \right\} \leq X(\theta t) + Y_0(t) \exp \left\{ t \lambda_0 V(Q_{R_2(t)}) \right\}
+ \sum_{n=1}^{\infty} \left( \mathbb{P}_0 \left\{ \tau_{Q_2(2^{n+1} t)} < 2t \right\} \right)^{1/q} \left( X(p \theta t) + Y_n(t) \exp \left\{ t \lambda_p V(Q_{R_2(2^{n+1} t)}) \right\} \right)^{1/p}
\]

where

\[
X(t) = \exp \left\{ t \sup_{|x| \leq \delta/2} |V_{\frac{\delta}{6},1}(x)| \right\}
\]

\[
Y_0(t) = \frac{48 R_2(t)^3}{\pi \delta^3} E_0 \exp \left\{ \sqrt{2} \delta T_1 \sup_{x \in Q_{R_2(t)}} |V_{\frac{\delta}{6},1}(x)| \right\}
\]

\[
Y_n(t) = \frac{48 R_2(2^n t)^3}{\pi \delta^3} E_0 \exp \left\{ \sqrt{2} \delta p T_1 \sup_{x \in Q_{R_2(2^n t)}} |V_{\frac{\delta}{6},1}(x)| \right\} \quad n = 1, 2, \ldots
\]
Using the classic fact that there is a constant \( C > 0 \) such that

\[
\mathbb{E}_0 \exp \{bT_1\} \leq \exp \{Cb^2\} \quad \forall b > 0
\]

we have

\[
\mathbb{E}_0 \exp \left\{ \sqrt{2\delta p\theta T_1} \sup_{x \in \mathcal{Q}_{R_2(2^n t)}} |\nabla_{\frac{8}{\pi},1}(x)| \right\} \leq \exp \left\{ 2\delta C(p\theta)^2 \left( \sup_{x \in \mathcal{Q}_{R_2(2^n t)}} |\nabla_{\frac{8}{\pi},1}(x)| \right)^2 \right\}
\]

\[
= \exp \left\{ o\left( \log(2^n t) \right)^2 \right\} \quad \text{a.s.} \quad (n \to \infty)
\]

where the last step follows from (2.2). Consequently,

\[
Y_n(t) = \exp \left\{ o\left( n^2 \right) \right\} \quad \text{a.s.} \quad (n \to \infty)
\]

Recall the classic fact that

\[
\mathbb{P} \left\{ \tau_{\mathcal{Q}_{R_2(2^n-1) t}} < 2t \right\} = \mathbb{P} \left\{ \max_{s \leq 2t} |B_s|_\infty \geq R_2(2^{n-1})t \right\}
\]

\[
= \mathbb{P} \left\{ \max_{s \leq 1} |B_s|_\infty \geq (2t)^{-1/2} R_2(2^{n-1})t \right\} \leq \exp \left\{ -C^2 \delta \frac{5l(2^{n-1})}{t^{4/3}} \right\}
\]

for some constant \( C > 0 \) independent of \( n \) and \( t \), where \(| \cdot |_\infty\) is the max-norm in \( \mathbb{R}^3 \). By Lemma 4.1 with \( k = 2 \) and with \( \theta \) being replaced by \( p\theta \),

\[
\lambda_{p\theta\nabla}(Q_{R_2(2^n t)}) = o\left( (2^n t)^2 l(2^n t)^{2/3} \right) \quad \text{a.s.} \quad (n \to \infty)
\]

Combining (4.2), (4.2), (4.2) we conclude that the right hand side of (4.2) is almost surely finite. Thus, we have established (4.2) conditioning on the event \( \{ \omega(B(0, \delta)) = 0 \} \). Therefore,

\[
\mathbb{P} \left\{ \mathbb{E}_0 \exp \left\{ \theta \int_0^t \nabla(B_s) ds \right\} < \infty \right\} \geq \mathbb{P} \left\{ \omega(B(0, \delta)) = 0 \right\} = \exp \left\{ -\frac{4}{3} \pi \delta^3 \right\}
\]

Since \( \delta \) can be arbitrarily small, we have completed the proof. \( \square \)

### 4.3 Upper bound for Theorem 1.2

Consider the decomposition

\[
\nabla(x) = \nabla_{1,1}(x) + \int_{\mathbb{R}^3} \frac{\alpha(|y - x|)}{|y - x|^2} \omega(dy) - \int_{\mathbb{R}^3} \frac{\alpha(|y|)}{|y|^2} dy \geq \nabla_{1,1}(x) - \int_{\mathbb{R}^3} \frac{\alpha(|y|)}{|y|^2} dy
\]

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where the notation $\overline{V}_{1,1}(x)$ is defined in section 2.1. We have that

\[ E_0 \exp \left\{ \theta \int_0^t \overline{V}(B_s)ds \right\} \geq E_0 \exp \left\{ \theta \int_0^t \overline{V}(B_s)ds; \tau_{B(0,t)} \geq t \right\} \geq \exp \left\{ - t \left( \sup_{|x| \leq t} |V_{1,1}(x)| + \int_{\mathbb{R}^3} \frac{\alpha(|y|)}{|y|^2} dy \right) \right\} \mathbb{P} \left\{ \max_{s \leq t} |B_s| \leq t \right\} \]

By (2.2) we have

\[ \lim_{t \to \infty} \frac{1}{t} \log E_0 \exp \left\{ \theta \int_0^t \overline{V}(B_s)ds \right\} \geq 0 \quad \text{a.s.} \]

To complete the proof of Theorem 1.2, therefore, all we need to show is that under the assumption (4.1),

\[ \limsup_{t \to \infty} t^{-\frac{\theta+1}{\theta} \lambda_\theta} \log E_0 \exp \left\{ \theta \int_0^t \overline{V}(B_s)ds \right\} \leq 0 \quad \text{a.s.} \]

conditioning on the event $\{ \omega(B(0, \delta)) = 0 \}$.

In the case $\frac{1}{24} < \theta < \frac{1}{16}$, the bound (4.2) holds when conditioned on $\{ \omega(B(0, \delta)) = 0 \}$.

By Lemma 4.1 with $k = 2$,

\[ \lambda_\theta(Q_{R_2(t)}) = o\left( t^2 l(t)^{2/3} \right) \quad \text{a.s.} \quad (t \to \infty) \]

The bound (4.2) can be replaced by

\[ Y_n(t) = \exp \left\{ o\left( \log(2^n t) \right)^2 \right\} \quad \text{a.s.} \quad n = 0, 1, \ldots \]

Combining these with the bound given in (4.2), (4.2), we have (4.3).

Now we consider the case $0 < \theta \leq \frac{1}{24}$ (so $k \geq 3$). The main reason we treat this setting separately for it includes the critical cases when $\theta = (8k)^{-1}$, which need some special care. Similar to (4.2), for any conjugate $p, q > 1$,

\[ E_0 \exp \left\{ \theta \int_0^t \overline{V}(B_s)ds \right\} \leq X(\theta t) + Y_0(t) \exp \left\{ t\lambda_\theta(Q_{R_k(t)}) \right\} \]

\[ + \sum_{n=1}^{\infty} \left( \mathbb{P} \left\{ \tau_{Q_{R_k(2^{n-1} \delta)} < 2t} \right\} \right)^{1/q} \left( X(p\theta t) + Y_n(t) \exp \left\{ t\lambda_{p\theta}(Q_{R_k(2^n \delta)}) \right\} \right)^{1/p} \]

where

\[ X(t) = \exp \left\{ t \sup_{|x| \leq \delta/2} |\overline{V}_{\frac{\delta}{2},1}(x)| \right\} \]

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\[
Y_0(t) = \frac{48 R_k(t)^3}{\pi \delta^3} \mathbb{E}_0 \exp \left\{ \sqrt{2 \delta \theta T_1} \sup_{x \in Q_{R_k(t)}} \left| V_{\delta,1}(x) \right| \right\}
\]
\[
Y_n(t) = \frac{48 R_k(2^n t)^3}{\pi \delta^3} \mathbb{E}_0 \exp \left\{ \sqrt{2 \delta \theta T_1} \sup_{x \in Q_{R_k(2^n t)}} \left| V_{\delta,1}(x) \right| \right\} \quad n = 1, 2, \cdots
\]

Similar to (4.2) and (4.2),
\[
Y_n(t) = \exp \left\{ o\left( (\log(2^n t))^2 \right) \right\} \quad a.s. \quad n = 0, 1, \cdots
\]
\[
\mathbb{P}_0 \left\{ \tau_{Q_{R_k(2^{n-1} t)}} < 2t \right\} \leq \exp \left\{ -C 2^n (2^n t)^{\frac{k+2}{k-2}} l(2^n t)^{\frac{4}{k-2}} \right\}
\]
Due to the possibility that \( \theta = (8k)^{-1} \), we can only make \( p \theta < (8(k-1))^{-1} \). So we may make \( (8k)^{-1} < p \theta < (8(k-1))^{-1} \). By the monotonicity of \( \lambda_{p\theta} V(D) \) in \( D \),
\[
\lambda_{p\theta} V(Q_{R_k(2^n t)}) \leq \lambda_{p\theta} V(Q_{R_{k-1}(2^n t)}) = o \left( \left( 2^n t \right)^{\frac{2}{k-2}} l(2^n t)^{\frac{2}{k-2}} \right) \quad a.s.
\]
where the second step follows from Lemma 4.1 with \( k \) being replaced by \( k - 1 \).

Summarizing the bounds we obtained, the infinite series on the right hand side of (4.3) is asymptotically (as \( t \to \infty \)) and almost surely bounded by
\[
C \sum_{n=1}^{\infty} \exp \left\{ -C^{-1} 2^n \right\}
\]
Finally, the requested (4.3)) follows from (4.3) and the fact (Lemma 4.1) that
\[
X(\theta t) + Y_0(t) \exp \left\{ t \lambda_{\theta} V(Q_{R_k(t)}) \right\} = \exp \left\{ o \left( t^{\frac{k+1}{k-1}} l(t) \frac{2}{k-2} \right) \right\} \quad a.s. \quad (t \to \infty)
\]
\[
\square
\]

### 4.4 Upper bound for Theorem 1.3

In view of (4.3), all we need is to show
\[
\liminf_{t \to \infty} t \frac{k+1}{k-1} l(t) \frac{2}{k-2} \log \mathbb{E}_0 \exp \left\{ \theta \int_0^t \nabla(B_s) ds \right\} \leq 0 \quad a.s.
\]
conditioning on the event \( \left\{ \omega(B(0, \delta)) = 0 \right\} \).

We prove (4.4) under the extra assumption that
\[
\int_1^{\infty} \frac{1}{t} \exp \left\{ -cl(t) \right\} < \infty
\]
\[
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\]
for some large constant $c > 0$, for otherwise we may consider $\tilde{l}(t) = \log \log t + l(t)$ instead of $l(t)$. Therefore, (4.2) is our new assumption.

Let $S_k(t)$ be given as in Lemma 4.2. We have that

$$\mathbb{E}_0 \exp \left\{ \theta \int_0^t V(B_s)ds \right\} \leq \mathbb{E}_0 \left[ \exp \left\{ \theta \int_0^t V(B_s)ds \right\}; \tau_{Q_{S_k(t)}} \geq 2t \right]$$

$$+ \left( \mathbb{P}_0 \{ \tau_{Q_{S_k(t)}} < 2t \} \right)^{1/q} \left( \mathbb{E}_0 \exp \left\{ p\theta \int_0^t V(B_s)ds \right\} \right)^{1/p}$$

where $p, q > 1$ are conjugate numbers.

In the case $\frac{1}{24} < \theta < \frac{1}{16}$ ($k = 2$), when can make $p$ close to 1 so $p\theta < \frac{1}{16}$. By the upper bound in Theorem 1.1 (with $\theta$ being replaced by $p\theta$ and $l(t) = (\log t)^2$)

$$\mathbb{E}_0 \exp \left\{ p\theta \int_0^t V(B_s)ds \right\} = \exp \left\{ o\left( t^3(\log t)^{4/3} \right) \right\} \text{ a.s. } (t \to \infty)$$

By the bound for Gaussian tail,

$$\mathbb{P}_0 \{ \tau_{Q_{S_2(t)}} < 2t \} \leq \exp \left\{ -Ct^{-1}S_2(t)^2 \right\} = \exp \left\{ -Ct^5l(t)^{-4/3} \right\}$$

Hence, the second term in (4.4) is negligible when $\frac{1}{24} < \theta < \frac{1}{16}$.

We now show that the same thing happens in the case when $0 < \theta \leq \frac{1}{24}$ ($k \geq 3$). In this case we can pick $p > 1$ such that $(8k)^{-1} < p\theta < (8(k-1))^{-1}$. By Theorem 1.2 (with $l(t) = (\log t)^2$ and $k$ being replaced by $k - 1$),

$$\mathbb{E}_0 \exp \left\{ p\theta \int_0^t V(B_s)ds \right\} = \exp \left\{ o\left( t^{k-2}(\log t)^{\frac{4}{3(k-2)}} \right) \right\} \text{ a.s. } (t \to \infty)$$

So our assertion follows from the Gaussian tail estimate

$$\mathbb{P}_0 \{ \tau_{Q_{S_2(t)}} < 2t \} \leq \exp \left\{ -Ct^{-1}S_k(t)^2 \right\} = \exp \left\{ -Ct^{2k-1}l(t)^{-\frac{4}{3(k-2)}} \right\}$$

Therefore, the problem has been reduced to the proof of

$$\liminf_{t \to \infty} t^{-\frac{k+1}{k-1}} l(t)^{\frac{2}{3(k-1)}} \log \mathbb{E}_0 \left[ \exp \left\{ \theta \int_0^t V(B_s)ds \right\}; \tau_{Q_{S_k(t)}} \geq 2t \right] \leq 0 \text{ a.s.}$$
conditioning on the event \( \{ \omega(B(0, \delta)) = 0 \} \).

By Lemma 2.7,

\[
\mathbb{E}_0 \left[ \exp \left\{ \theta \int_0^t V(B_s) ds \right\}; \ \tau_{Q_S(t)} \geq 2t \right] \leq \exp \left\{ \theta t \sup_{|x| \leq \delta/2} |V_{\delta/2}(x)| \right\} \\
+ \frac{6|Q_{S_k(t)}|}{\pi \delta^3} \mathbb{E}_0 \exp \left\{ \sqrt{2\delta T_1 \theta} \sup_{x \in Q_{S_k(t)}} |V_{\delta/2}(x)| \right\} \exp \left\{ t \lambda_{V}(Q_{S_k(t)}) \right\}
\]

\[
= \exp\{O(t)\} + \exp \left\{ \left( o(\log S_k(t)) \right)^2 \right\} \exp \left\{ t \lambda_{V}(Q_{S_k(t)}) \right\} \quad \text{a.s.} \quad (t \to \infty)
\]

where the last step follows from (2.2).

Finally, the requested (4.4) follows from Lemma 4.2. □

## 5 Hardy inequality

The essential reason behind the main theorems in this paper is the Hardy’s inequality. Its special form in \( \mathbb{R}^3 \) gives

\[
\int_{\mathbb{R}^3} \frac{f^2(x)}{|x|^2} dx \leq 4 \int_{\mathbb{R}^3} |\nabla f(x)|^2 dx \quad f \in W^{1,2}(\mathbb{R}^3)
\]

with 4 being the best constant in the sense that for any \( \epsilon > 0 \) one can find a function \( f_\epsilon \in W^{1,2}(\mathbb{R}^3) \) with compact support such that

\[
\int_{\mathbb{R}^3} \frac{f_\epsilon^2(x)}{|x|^2} dx > (4 - \epsilon) \int_{\mathbb{R}^3} |\nabla f_\epsilon(x)|^2 dx
\]

See, for example, [16] and [20] for an overview on Hardy inequality.

**Lemma 5.1** For any \( \theta > 0 \),

\[
\sup_{g \in \mathcal{F}_3} \left\{ \theta \int_{\mathbb{R}^3} \frac{g^2(x)}{|x|^2} dx - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla g(x)|^2 dx \right\} = \begin{cases} 0 & \text{if } \theta \leq \frac{1}{8} \\ \infty & \text{if } \theta > \frac{1}{8} \end{cases}
\]

**Proof:** By Hardy’s inequality, the left hand side of (5.1) is no-positive when \( \theta < 1/8 \). On the other hand, it is no less than

\[
-\frac{1}{2} \inf_{g \in \mathcal{F}_3} \int_{\mathbb{R}^3} |\nabla g(x)|^2 dx
\]

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which is equal to zero. Thus, for $\theta \leq 1/8$,

$$
\sup_{g \in \mathcal{F}_3} \left\{ \theta \int_{\mathbb{R}^3} \frac{g^2(x)}{|x|^2} \, dx - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla g(x)|^2 \, dx \right\} = 0
$$

Assume $\theta > 1/8$. By the optionality of Hardy’s inequality described in (5),

$$
M(\theta) \equiv \sup_{g \in \mathcal{F}_3} \left\{ \theta \int_{\mathbb{R}^3} \frac{g^2(x)}{|x|^2} \, dx - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla g(x)|^2 \, dx \right\} > 0
$$

Given $a > 0$, the substitution $g(x) = a^{3/2} f(ax)$ leads to $M(\theta) = a^2 M(\theta)$. So $M(\theta) = \infty$. □

References


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