Exercise 3.3
Proof:
\[ P(V(t) = 1) = P(V(t) = 1|V(0) = 0)P(V(0) = 0) + P(V(t) = 1|V(0) = 1)P(V(0) = 1) = p_{01}(t)(1 - \pi) + p_{11}(t)\pi = (\pi - \pi e^{-\tau t})(1 - \pi) + (\pi + (1 - \pi)e^{-\tau t})\pi = \pi \]

Problem 3.3
Proof: For 0 < s < t,
\[ E[V(s)V(t)] = E[V(s)V(t)|V(s) = 0]P(V(s) = 0) + E[V(s)V(t)|V(s) = 1]P(V(s) = 1) = \pi p_{11}(t - s) = \pi - \pi p_{10}(t - s) \]

hence,
\[ \text{Cov}[V(s)V(t)] = E[V(s)V(t)] - E[V(s)]E[V(t)] = \pi - \pi p_{10}(t - s) - \pi^2 = \pi(1 - \pi) - \pi(1 - \pi)(1 - e^{-(\alpha + \beta)(t - s)}) = \pi(1 - \pi)e^{-(\alpha + \beta)(t - s)} \]

Problem 3.4
Solution: Let \( \pi = \frac{\alpha}{\alpha + \beta} \) and \( \tau = \alpha + \beta \). We have
\[ E[S(t)] = \int_0^t E[V(u)]du = \int_0^t p_{01}(u)du = \int_0^t (\pi - \pi e^{-\tau u})du = \pi t - \frac{\pi}{\tau}(1 - e^{-\tau t}) \]
Exercise 4.2
Solution: We have \( \theta_0 = 1 \) and for \( 1 \leq n \leq N \),
\[
\theta_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} = \frac{\alpha^n N(N - 1) \cdots (N - n + 1)}{\beta^n n!} = \binom{N}{n} \left( \frac{\alpha}{\beta} \right)^n.
\]
therefore, we have
\[
\pi_n = \frac{\theta_n}{\sum_{m=0}^{N} \theta_m} = \frac{\binom{N}{n} \left( \frac{\alpha}{\beta} \right)^n}{\sum_{m=0}^{N} \binom{N}{m} \left( \frac{\alpha}{\beta} \right)^m} = \binom{N}{n} \left( \frac{\alpha}{\alpha + \beta} \right)^n \left( \frac{\beta}{\alpha + \beta} \right)^{N-n}.
\]

Problem 4.3
Solution: Since the operating time for each machine is i.i.d. exponentially distributed and the repair time is also exponentially distributed, this problem can be modeled as a birth-death process with state space \( \{0, 1, 2, 3, 4, 5\} \) where state \( k \) (\( 0 \leq k \leq 5 \)) means that \( k \) machine is broken at that moment. Since the operating time for each machine is exponentially distributed with parameter 0.2, we know under the condition that no machine is broken right now, the first time for a machine breaks is an exponential time with parameter \( 0.2 \times 5 = 1 \), i.e. \( \lambda_0 = 1 \). Similar argument leads to the fact that \( \lambda_k = 0.2(5 - k) \). As to \( \mu_k \), they are all equal to 0.5. Hence, we have the infinitesimal generator as
\[
\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 \\
0 & -1 & 1 & & & \\
1 & 0.5 & -1.3 & 0.8 & & \\
2 & & 0.5 & -1.1 & 0.6 & \\
3 & & & 0.5 & -0.9 & 0.4 \\
4 & & & & 0.5 & -0.7 & 0.2 \\
5 & & & & & 0.5 & -0.5
\end{pmatrix}
\]

Therefore, \( \theta_0 = 1, \theta_1 = 2, \theta_2 = \theta_1 \cdot \frac{\lambda_1}{\mu_2} = \frac{16}{5}, \theta_3 = \theta_2 \cdot \frac{\lambda_2}{\mu_3} = \frac{96}{25} \).
\[
\theta_4 = \theta_3 \cdot \frac{\lambda_3}{\mu_4} = \frac{384}{125}, \theta_5 = \theta_4 \cdot \frac{\lambda_4}{\mu_5} = \frac{768}{625}.
\]
Hence we have ‘The fraction of time which the repairman is idle’ \( = \pi_0 = \frac{\theta_0}{\theta_0 + \theta_1 + \cdots + \theta_5} \) \( \approx 0.0697 \) \( \Box \).