MATH424 HOMEWORK ANSWER

Homework 10. (Chapter 6)

Exercise 2.1
Solution: We have \( P_3(t) = e^{-\mu_3 t} = e^{-5t} \),

\[
P_2(t) = \mu_3 \left( A_{22} e^{-\mu_2 t} + A_{32} e^{-\mu_3 t} \right)
= 5 \left( \frac{1}{\mu_3 - \mu_2} e^{-\mu_2 t} + \frac{1}{\mu_2 - \mu_3} e^{-\mu_3 t} \right)
= \frac{5}{3} \left( e^{-2t} - e^{-5t} \right),
\]

\[
P_1(t) = \mu_2 \mu_3 \left( A_{11} e^{-\mu_1 t} + A_{21} e^{-\mu_2 t} + A_{31} e^{-\mu_3 t} \right)
= \mu_2 \mu_3 \left( \frac{1}{(\mu_2 - \mu_1)(\mu_3 - \mu_1)} e^{-\mu_1 t} + \frac{1}{(\mu_3 - \mu_2)(\mu_1 - \mu_2)} e^{-\mu_2 t} + \frac{1}{(\mu_2 - \mu_3)(\mu_1 - \mu_3)} e^{-\mu_3 t} \right)
= 10 \left( \frac{1}{2} e^{-3t} + \frac{1}{6} e^{-2t} + \frac{1}{6} e^{-5t} \right)
\]

and \( P_0(t) = 1 - P_1(t) - P_2(t) - P_3(t) \). \qed

Exercise 2.3
Solution: We have \( P_3(t) = e^{-\mu_3 t} = e^{-t} \),

\[
P_2(t) = \mu_3 \left( A_{22} e^{-\mu_2 t} + A_{32} e^{-\mu_3 t} \right)
= 5 \left( \frac{1}{\mu_3 - \mu_2} e^{-\mu_2 t} + \frac{1}{\mu_2 - \mu_3} e^{-\mu_3 t} \right)
= -e^{-2t} + e^{-t}
\]

\[
P_1(t) = \mu_2 \mu_3 \left( A_{11} e^{-\mu_1 t} + A_{21} e^{-\mu_2 t} + A_{31} e^{-\mu_3 t} \right)
= \mu_2 \mu_3 \left( \frac{1}{(\mu_2 - \mu_1)(\mu_3 - \mu_1)} e^{-\mu_1 t} + \frac{1}{(\mu_3 - \mu_2)(\mu_1 - \mu_2)} e^{-\mu_2 t} + \frac{1}{(\mu_2 - \mu_3)(\mu_1 - \mu_3)} e^{-\mu_3 t} \right)
= 2 \left( \frac{1}{2} e^{-3t} - e^{-2t} + \frac{1}{2} e^{-t} \right)
\]

and \( P_0(t) = 1 - P_1(t) - P_2(t) - P_3(t) \). \qed

Problem 2.2
Solution: Let’s define a new process \( Y(t) \equiv N - X(t) \). Since \( X(t) \) is a pure death process with constant death rate \( \theta \), \( Y(t) \) can be viewed as a truncated Poisson process below \( N \), namely, \( Y(t) \) has the same distribution
as $Z(t) \wedge N$ where $Z(t)$ is a Poisson process with parameter $\theta$. Therefore, for $1 \leq n \leq N$,

$$P_n(t) = P(X(t) = n) = P(N - Z(t) \wedge N = n) = P(Z(t) \wedge N = N - n) = P(Z(t) = N - n) = \frac{(\theta t)^{N-n}}{(N-n)!}e^{-\theta t}$$

and for $n = 0$,

$$P_n(t) = P(X(t) = 0) = P(Z(t) \wedge N = N) = P(Z(t) \geq N) = 1 - \sum_{k=0}^{N-1} \frac{(\theta t)^k}{k!}e^{-\theta t}.$$  \[ \Box \]

**Problem 2.6**

**Proof:** (a) We know $S_k$ is an exponential time with parameter $\alpha k$, therefore

$$E(T) = E(S_1 + \cdots + S_N) = E(S_1) + \cdots + E(S_N) = \frac{1}{\alpha \cdot N} + \cdots + \frac{1}{\alpha \cdot 1} = \frac{1}{\alpha} \left( \frac{1}{N} + \frac{1}{N-1} + \cdots + \frac{1}{1} \right).$$

(b).

$$E(T) = \int_0^\infty P(T > t)dt = \int_0^\infty (1 - F_T(t))dt = \int_0^\infty \left( 1 - \left( 1 - e^{-\alpha t} \right)^N \right) dt \left( \text{Let } y = 1 - e^{-\alpha t}, \ then \ dt = \frac{1}{\alpha(1-y)}dy \right) = \int_0^1 \frac{1 - y^N}{\alpha(1-y)}dy = \frac{1}{\alpha} \int_0^1 \left( 1 + y + \cdots + y^{N-1} \right) dy = \frac{1}{\alpha} \left( \frac{1}{N} + \frac{1}{N-1} + \cdots + \frac{1}{1} \right). \quad \Box$$
Exercise 3.2
Solution: Notice that the treatment time only depends on the number of patients at that time but not how long the patient has been waiting, therefore, \( N(t) \) has Markovian property. Furthermore, since there is only one doctor who can only serve one patient at a time and the arrival of patients is a Poisson process. Therefore, \( N(t) \) might be modeled as a birth and death process. The necessary assumption should be: for any \( t \),

\[
P( \text{Particular patient finish the treatment in } [t, t + h]|k \text{ patients at time } t) = \frac{1}{m_k}h + o(h). \quad \Box
\]

Problem 3.1
Solution: First of all, let’s show that \( X(t) \) is a Markov Chain. To show this, for \( s, t > 0, n \in \mathbb{N} \) and any \( 0 \leq s_1 < s_2 < \cdots < s_n = s \), we have

\[
P(X(s + t) = j|X(s_1) = i_1, \ldots, X(s_n) = i_n) = P(\xi_{N(s+t)} = j|\xi_{N(s_1)} = i_1, \ldots, \xi_{N(s_n)} = i_n) = P(\xi_{N(s+t)} = j|\xi_{N(s_n)} = i_n) \quad (0.1)
\]

where (0.1) holds due to the fact that \( 0 \leq N(s_1) < N(s_2) < \cdots < N(s_n) \) and \( \xi_n \) is a Markov Chain. Hence, we have \( X(t) \) is a two state Markov Chain, too.

Secondly, we want to show \( X(t) \) is a time homogeneous Markov Chain:

\[
P(X(s + t) = i|X(s) = j) = P(\xi_{N(s+t)} = i|\xi_{N(s)} = j) = P(\xi_{N(s+t)-N(s)} = i|\xi_0 = j) \quad (0.2)
\]

where (0.2) holds because \( \xi_n \) is a time homogeneous Markov Chain and \( \xi_n \) and \( N(t) \) are independent; (0.3) holds since \( N(t) \) is time homogeneous.
Finally, we will show $X(t)$ is a birth and death process. In fact,

\[ P(X(t+h) = 1 | X(t) = 0) = P(X(h) = 1) | X(0) = 0 \]
\[ = P(\xi_{N(h)} = 1 | \xi_0 = 0) \]
\[ = \sum_{k=1}^{\infty} P(\xi_k = 1 | \xi_0 = 0) P(N(h) = k) \]
\[ = \sum_{k=1}^{\infty} P(\xi_k = 1 | \xi_0 = 0) \left( \frac{(\lambda h)^k}{k!} e^{-\lambda h} \right) \]
\[ = \sum_{k=1}^{\infty} P(\xi_k = 1 | \xi_0 = 0) \left( \frac{(\lambda h)^k}{k!} \right) (1 - \lambda h + o(h)) \]
\[ = P(\xi_1 = 1 | \xi_0 = 0) \lambda h + o(h) \]
\[ = \lambda h + o(h); \]

\[ P(X(t+h) = 0 | X(t) = 0) = P(X(h) = 0) | X(0) = 0 \]
\[ = P(\xi_{N(h)} = 0 | \xi_0 = 0) \]
\[ = \sum_{k=0}^{\infty} P(\xi_k = 0 | \xi_0 = 0) P(N(h) = k) \]
\[ = \sum_{k=0}^{\infty} P(\xi_k = 0 | \xi_0 = 0) \left( \frac{(\lambda h)^k}{k!} e^{-\lambda h} \right) \]
\[ = \sum_{k=0}^{\infty} P(\xi_k = 0 | \xi_0 = 0) \left( \frac{(\lambda h)^k}{k!} \right) (1 - \lambda h + o(h)) \]
\[ = P(\xi_0 = 0 | \xi_0 = 0) (1 - \lambda h) + o(h) \]
\[ = 1 - \lambda h + o(h); \]

\[ P(X(t+h) = 0 | X(t) = 1) = P(X(h) = 0) | X(0) = 1 \]
\[ = P(\xi_{N(h)} = 0 | \xi_0 = 1) \]
\[ = \sum_{k=1}^{\infty} P(\xi_k = 0 | \xi_0 = 1) P(N(h) = k) \]
\[ = \sum_{k=1}^{\infty} P(\xi_k = 0 | \xi_0 = 1) \left( \frac{(\lambda h)^k}{k!} e^{-\lambda h} \right) \]
\begin{align*}
\sum_{k=1}^{\infty} P(\xi_k = 0|\xi_0 = 1) \frac{(\lambda h)^k}{k!} (1 - \lambda h + o(h)) \\
= P(\xi_1 = 0|\xi_0 = 1) \lambda h + o(h) \\
= (1 - \alpha) \lambda h + o(h);
\end{align*}

and

\begin{align*}
P(X(t+h) = 1|X(t) = 1) \\
= P(X(h) = 1)|X(0) = 1 \\
= P(\xi_N(h) = 1|\xi_0 = 1) \\
= \sum_{k=0}^{\infty} P(\xi_k = 1|\xi_0 = 1) P(N(h) = k) \\
= \sum_{k=0}^{\infty} P(\xi_k = 1|\xi_0 = 1) \frac{(\lambda h)^k}{k!} e^{-\lambda h} \\
= \sum_{k=0}^{\infty} P(\xi_k = 1|\xi_0 = 1) \frac{(\lambda h)^k}{k!} (1 - \lambda h + o(h)) \\
= P(\xi_0 = 1|\xi_0 = 1)(1 - \lambda h) + P(\xi_1 = 1|\xi_0 = 1) \lambda h + o(h) \\
= 1 - \lambda h + \alpha \lambda h + o(h) \\
= 1 - (1 - \alpha) \lambda h + o(h).
\end{align*}

hence, we obtain that $X(t)$ is a two state birth and death process with parameter $\lambda_0 = \lambda$ and $\mu_1 = (1 - \alpha) \lambda$. \qed

**Problem 3.2**

**Solution:** Since we want to model the position of the beetle at time $t$ as a birth and death process and the mean staying time at position 0 and $N$ are $m_0$ and $m_N$, respectively, the assumption at the ends 0 and $N$ should be:

\begin{align*}
P(\text{The beetle move right to 1 in } [t, t+h]) \text{ the beetle at position 0 at time } t) \\
= \frac{1}{m_0} h + o(h) \\
P(\text{The beetle stays at 0 in } [t, t+h]) \text{ the beetle at position 0 at time } t) \\
= 1 - \frac{1}{m_0} h + o(h)
\end{align*}
\[ P(\text{The beetle move left to } N - 1 \text{ in } [t, t + h] \text{ the beetle at position } N \text{ at time } t) \]
\[ = \frac{1}{m_N} h + o(h) \]

\[ P(\text{The beetle stays at } N \text{ in } [t, t + h] \text{ the beetle at position } N \text{ at time } t) \]
\[ = 1 - \frac{1}{m_N} h + o(h) \]

\[ \square \]