EXACT CONVERGENCE RATES FOR THE DISTRIBUTION OF THE PARTICLES IN BRANCHING RANDOM WALKS

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The exact convergence rates of the particle distributions in the supercritical branching random walks and supercritical branching Wiener processes are obtained and a conjecture of Révész (1994) is confirmed.

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1. Introduction

Consider a branching particle system starting from one ancestor at the origin in a d-dimensional space. Independently, each particle moves to a new site after one time unit since its birth, gives birth to a random number of offsprings and die. The same procedure is repeated by all generations. Throughout the migration is governed either by a d-dimensional simple symmetric random walk or by a d-dimensional Wiener process, and the reproduction by a Galton-Watson tree whose offspring distribution has the mean $m > 1$ and finite variance. This model is called *branching random walk* (when migration is executed by random walk), or *branching Wiener process* (when migration is executed by Wiener process). Under our assumptions, the random sequence $\{B(t)\}_{t\geq 0}$ with $B(t)$ being given as the total population in generation t ($t \geq 0$) is a supercritical Branching chain. It is well known (see, c.f., Athreya-Ney (1972)) that

$$\lim_{t\to\infty} \frac{B(t)}{m^t} = B \qquad a.s. \tag{1.1}$$

for some random variable B which is not constantly zero.

In addition to their obvious background in the study of population growth and migration, the models of branching random walks (Wiener processes) had their origins in the theory of cascade processes. The study of branching random walks as a probability problem was initiated by Kolmogorov (1941) (The reader is referred to a survey by Ney

(1991) for historical account and for general information of this field). A central limit theorem conjectured by Harris (1963, p.75) states that

$$\frac{1}{m^T}\sum_{y\le x\sqrt{T}}\lambda(y,T)\longrightarrow BG(x)\qquad a.s. \tag{1.2}$$

where $\lambda(x,T)$ is the population of the particles located at x at time T and, $G(x)$ is the d-dimensional normal distribution function attracting the migration random walk through the classic central limit theorem. See, e.g., Stam (1966), Asmussen-Kaplan (1976), Athreya-Kaplan (1978), Klebaner (1982), Joffe (1985), Biggins (1990), Bramson-Ney-Tao (1992) and Révész (1994) for the developments on this subject. Concerning the speed of above convergence, Révész (1994) proves that for each $\epsilon>0$,

$$T^{1/2-\epsilon}\Big(\frac{1}{m^T}\sum_{y\le x\sqrt{T}}\lambda(y,T)-BG(x)\Big)\longrightarrow 0\qquad \text{a.s.} \tag{1.3}$$

Like the classic central limit theorem, the central limit theorem for branching random walks yields its local version (see also Watanabe (1965), Athreya-Kang (1998) for the local central limit theorems for a variety of branching Markov processes). In the case of branching random walk, Révész (1994) shows that

$$T^{1-\epsilon}\Big(\frac{1}{2}\Big(\frac{4\pi T}{d}\Big)^{d/2}\frac{\lambda(0,2T)}{m^{2T}}-B\Big)\longrightarrow 0\qquad \text{a.s.} \tag{1.4}$$

Naturally, one wonders if (1.3) and (1.4) suggest the exact rates of convergence. Indeed, a counterpart of (1.4) given in Theorem 4.9 of Révész (1994) says that for each $C>0$, there is a $\delta=\delta(C)>0$ such that

$$P\Big\{\Big|\frac{1}{2}\Big(\frac{4\pi T}{d}\Big)^{d/2}\frac{\lambda(0,2T)}{m^{2T}}-B\Big|\ge\frac{C}{T}\Big\}\ge\delta \tag{1.5}$$

for sufficiently large T. This observation makes him conjecture (p.79, Révész (1994)) that the sequence

$$T\Big(\frac{1}{2}\Big(\frac{4\pi T}{d}\Big)^{d/2}\frac{\lambda(0,2T)}{m^{2T}}-B\Big)\qquad T=1,2,\cdots \tag{1.6}$$

weakly converges to some non-degenerate random variable as $T\to\infty$.

This paper is to find the exact convergence rates for these limit theorems, and to settle the conjecture raised by Révész in particular. Instead of the weak convergence proposed by Révész, we shall prove his conjecture in terms of almost sure convergence as well as L_2-convergence. Our tools are some decompositions given in Révész (1994) and martingale approximations.

The rest of the paper is organized as following: In section 2, we give our results (Theorem 2.1, Theorem 2.2 and Corollary 2.3) for branching random walks. In section

3, we point out their analogues (Theorem 3.1 and Theorem 3.2) in the case of branching Wiener processes. Theorem 2.1, Theorem 2.2 and Corollary 2.3 are proved in section 4. Due to similarity, only a sketch is given to the proofs of Theorem 3.1 and Theorem 3.2 in section 5.

The following notations and assumptions will be kept throughout the article. For $x = (x_1, \cdots, x_d), y = (x_1, \cdots, y_d) \in \mathbf{R}^d$, $x \cdot y$ and $||x||$ will be used, respectively, for the inner product between x, y and for the Euclidean norm of x. The partial order "$x \leq y$" is defined by the relation $x_1 \leq y_1, \cdots, x_d \leq y_d$. Given a measurable $A \subset \mathbf{R}^d$, $|A|$ denotes its Lebesgue measure. Write

$$\Phi_d(x) = \Big(\frac{1}{2\pi}\Big)^{d/2} \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_d} \exp\Big\{ -\frac{||y||^2}{2}\Big\} dy$$

and let $\Phi(x) = \Phi_1(x)$.

We use the non-negative integer valued random variable Z to represent the distribution of the number of children of each individual in our particle system and assume

$$m \equiv EZ > 1 \quad \text{and} \quad \sigma^2 \equiv Var(Z) < \infty. \tag{1.7}$$

2. Results for branching random walks

We begin with a formal definition of the local population $\lambda(x,t)$. Let $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_d$ be the orthogonal unit vectors in the d-dimensional lattice $\mathbf{Z}^d$ and let X be a $\mathbf{Z}^d$-valued random variable independent of Z with

$$P\{X = \mathbf{e}_j\} = \frac{1}{2d} \qquad j = 1, 2, \cdots d,$$

and let

$$\Big\{\big(X(x,t,k), Z(x,t,k)\big); \quad x \in \mathbf{Z}^d,\ t = 0, 1, 2, \cdots, k = 1, 2, \cdots\Big\}$$

be an array of i.i.d. random vector with

$$\big(X(\mathbf{0}, 0, 1), Z(\mathbf{0}, 0, 1)\big) = (X, Z).$$

Intuitively, we coordinate each individual in our particle system by the 3-tuple (x, t, k), where x represents his birth site, t represents his generation (so the original ancestor belongs to generation 0) and k is his order number as one of the members who were born at x in his generation. For given individual (x, t, k), $X(x, t, k)$ is interpreted as the migration made by him, and $Z(x, t, k)$ is the number of his children. The local population $\lambda(x, t)$ at $x \in \mathbf{Z}^d$ in the generation t is defined as following:

$$\lambda(x, 0) = \begin{cases} 1 & \text{if } x = \mathbf{0}, \\ 0 & \text{if } x \neq \mathbf{0}, \end{cases}$$

$$\lambda(x,t) = \sum_{y\in\mathbf{Z}^d} \sum_{k=1}^{\lambda(y,t-1)} I_{x-y}\big(X(y,t-1,k)\big) Z(y,t-1,k)$$

where $x = (x_1,\cdots,x_d) \in \mathbf{Z}^d$ and $t = 1,2,\cdots$. Clearly, $\lambda(x,t) = 0$ if $t \not\equiv x_1 + \cdots + x_d$ mod (2).

Write

$$B(t) = \sum_{x\in\mathbf{Z}^d} \lambda(x,t) \qquad t = 0,1,2,\cdots.$$

Then $\{B(t)\}_{t\ge 0}$ is a supercritical Branching chain starting with $B(0) = 1$ and having the offspring distribution $\mathcal{L}(Z)$ (see, e.g., Athreya-Ney (1972) for the detail of branching chains). It is well known that when $m > 1$, $\{B(t)\}_{t\ge 0}$ survives with positive probability.

Let

$$\mathcal{F}(t) = \mathcal{F}\{\lambda(x,s);\quad x\in\mathbf{Z}^d,\ s = 0,1,\cdots,t\}$$

be the σ-algebra generated by the array

$$\{\lambda(x,s);\quad x\in\mathbf{Z}^d,\ s = 0,1,\cdots,t\}.$$

Theorem 2.1. *There exist a real random variable M and a $\mathbf{R}^d$-valued random variable N such that for each $x = (x_1,\cdots,x_d) \in \mathbf{Z}^d$,*

$$T\Big[\frac{1}{2}\Big(\frac{2\pi T}{d}\Big)^{d/2}\frac{\lambda(x,T)}{m^T} - B\exp\Big\{-\frac{d||x||^2}{2T}\Big\}\Big] \longrightarrow d\big(\frac{1}{2}M + x\cdot N\big), \tag{2.1}$$

almost surely as well as in L_2-norm, as $T\to\infty$ with $T \equiv x_1 + \cdots + x_d$ mod(2), where the random variable B is given in (1.1).

Besides, the random variables M and N satisfy the following:

$$EM = 0 \quad and \quad EM^2 = \frac{4(m^2+\sigma^2)}{d(m-1)^3}, \tag{2.2}$$

$$EN = \mathbf{0} \quad and \quad \mathbf{cov}(N,N) = \frac{m^2+\sigma^2}{d(m-1)^2}\mathbf{I}_d, \tag{2.3}$$

$$\big(B,M,N\big) \overset{\mathrm{d}}{=} \big(B,M,-N\big), \tag{2.4}$$

$$E\big[M|\mathcal{F}(t)\big] = t\frac{B(t)}{m^t} - \frac{1}{m^t}\sum_{y\in\mathbf{Z}^d} ||y||^2\lambda(y,t) \qquad t = 0,1,\cdots, \tag{2.5}$$

$$E\big[N|\mathcal{F}(t)\big] = \frac{1}{m^t}\sum_{y\in\mathbf{Z}^d} y\lambda(y,t) \qquad t=0,1,\cdots, \tag{2.6}$$

where $\mathbf{I}_d$ is the $d\times d$ identity matrix.

Further, if $\{(B_k, M_k, N_k)\}_{k\geq 1}$ are independent copies of (B, M, N) and if they are independent of (X, Z) then

$$\Big(B,\ M,\ (N-BX)\Big) \overset{\mathrm{d}}{=} \frac{1}{m}\sum_{k=1}^{Z}\Big(B_k,\ \big(M_k - 2X\cdot N_k\big),\ N_k\Big). \tag{2.7}$$

Theorem 2.2. *For each $x\in\mathbf{Z}^d$,*

$$\sqrt{T}\Big[\frac{1}{m^T}\sum_{y\leq x\sqrt{T}}\lambda(y,T) - BP\big\{S_T\leq x\sqrt{T}\big\}\Big] \longrightarrow -\nabla\Phi_d(\sqrt{d}x)\cdot N \tag{2.8}$$

almost surely as well as in L_2-norm, provided $T\to\infty$, where B is given in (1.1), N is given in Theorem 2.1 and $\{S_t\}$ is the symmetric simple random walk generated by X.

Remark. Taking $x=0$ in (2.1) we see that the sequence in (1.6) converges almost surely as well as in L_2-norm. So the conjecture made by Révész (1994) is proved. From (1.5) one can also see that the random variable M in Theorem 2.1 is unbounded. By Proposition 1.2.5 of Lawler (1991),

$$P^T(x)\equiv P\{S_T=x\} = 2\Big(\frac{d}{2\pi T}\Big)^{d/2}\exp\Big\{-\frac{d||x||^2}{2T}\Big\} + O\big(T^{-2-d/2}\big)$$

as $T\to\infty$ with $T\equiv x_1+\cdots+x_d$ mod (2). Therefore (2.1) is equivalent to

$$T^{1+d/2}\left[\frac{\lambda(x,T)}{m^T} - BP^T(x)\right] \longrightarrow d\Big(\frac{d}{2\pi}\Big)^{d/2}\big(M+2x\cdot N\big). \tag{2.9}$$

Nevertheless, $P\big\{S_T\leq x\sqrt{T}\big\}$ in Theorem 2.2 can not be replaced by $\Phi_d(\sqrt{d}x)$. Indeed, we have

Corollary 2.3. *For each $x\in\mathbf{Z}^d$,*

$$\begin{aligned}&\sqrt{T}\Big(\frac{1}{m^T}\sum_{y\leq x\sqrt{T}}\lambda(y,T) - B\Phi_d(\sqrt{d}x)\Big)\\ &= \nabla\Phi_d(\sqrt{d}x)\cdot\Big(B\mathbf{F}(\sqrt{T}x) - 2N\Big) + o(1) \qquad (T\to\infty)\end{aligned} \tag{2.10}$$

almost surely as well as in L_2-norm, where B is given in (1.1), N is given in Theorem 2.1, $\mathbf{F}(x) = \big(f(x_1),\cdots,f(x_d)\big)$ and $f\colon(-\infty,\infty)\longrightarrow\Big(-\frac{1}{2},\frac{1}{2}\Big)$ is a periodic function with period 1, $f(k)=0$ $(k=0,\pm1,\pm2,\cdots)$ and

$$f(\theta) = \frac{1-2\theta}{2} \qquad 0<\theta<1.$$

Corollary 2.3 shows that asymptotically, the sequence

$$\sqrt{T}\Big(\frac{1}{m^T}\sum_{y\le x\sqrt{T}}\lambda(y,T)-B\Phi_d(\sqrt{d}x)\Big)\qquad T=1,2,\cdots$$

oscillates in a finite random interval. So the exact rate for the global central limit theorem is established.

3. Results for branching Wiener processes

The construction of the branching Wiener process is similar. Let $W(t)$ be a standard d-dimensional Wiener process independent of Z and write $W=W(1)$. Let

$$\big\{\big(W(x,t,k),Z(x,t,k)\big);\quad x\in\mathbf{R}^d,\ t=0,1,2,\cdots,k=1,2,\cdots\big\}$$

be a set of i.i.d. random vectors such that

$$\big(W(\mathbf{0},0,1),Z(\mathbf{0},0,1)\big)=(W,Z).$$

Define

$$\lambda(x,0)=\begin{cases}1 & \text{if } x=\mathbf{0}\\ 0 & \text{if } x\neq\mathbf{0},\end{cases}$$

$$\lambda(x,t)=\sum_{y\in\mathbf{R}^d}\sum_{k=1}^{\lambda(y,t-1)}I_x\big(y+W(y,t-1,k)\big)Z(y,t-1,k)$$

where $x\in\mathbf{R}^d$ and $t=1,2,\cdots$. Clearly, $\lambda(x,t)=0$ for all but finitely many $x\in\mathbf{R}^d$. Define the random measure

$$\psi(A,t)=\sum_{x\in A}\lambda(x,t)=\sum_{y\in\mathbf{R}^d}\sum_{k=1}^{\lambda(y,t-1)}I_A\big(y+W(y,t-1,k)\big)Z(y,t-1,k)$$

for all measurable $A\subset\mathbf{R}^d$. Let

$$\begin{aligned}\mathcal{F}(t)&=\mathcal{F}\{\lambda(x,s);\quad x\in\mathbf{R}^d,\ s=0,1,\cdots,t\}\\ &=\mathcal{F}\{\psi(A,s);\quad A\subset\mathbf{R}^d,\ s=0,1,\cdots,t\}\end{aligned}$$

be the σ-algebra generated by

$$\{\lambda(x,s);\quad x\in\mathbf{Z}^d,\ s=0,1,\cdots,t\}.$$

Theorem 3.1. *There exist a real random variable M and a $\mathbf{R}^d$-valued random variable N such that for each $A\subset\mathbf{R}^d$ with $|A|>0$ and $\int_A||x||dx<+\infty$,*

$$T\Big[(2\pi T)^{d/2}\frac{\psi(A,T)}{m^T}-B\int_A\exp\Big\{-\frac{||x||^2}{2T}\Big\}dx\Big]\longrightarrow|A|\big(\frac{1}{2}M+\bar{x}_A\cdot N\big)\qquad(3.1)$$

almost surely as well as in L_2-norm, as $T \to \infty$, where B is given in (1.1) and

$$\bar{x}_A = \frac{1}{|A|}\int_A x dx.$$

Besides, the random variables M and N satisfy the following:

$$EM = 0 \quad and \quad EM^2 = \frac{2d(m^2+\sigma^2)(m+1)}{(m-1)^3}, \tag{3.2}$$

$$EN = \mathbf{0} \quad and \quad \mathbf{cov}\big(N,N\big) = \frac{(m^2+\sigma^2)}{(m-1)^2}\mathbf{I}_d, \tag{3.3}$$

$$\big(B, M, N\big) \stackrel{\mathrm{d}}{=} \big(B, M, -N\big) \tag{3.4}$$

$$E\big[M|\mathcal{F}(t)\big] = dt\frac{B(t)}{m^t} - \frac{1}{m^t}\int ||y||^2\psi(dy,t) \qquad t = 0, 1, \cdots, \tag{3.5}$$

$$E\big[N|\mathcal{F}(t)\big] = \frac{1}{m^t}\int y\psi(dy,t) \qquad t = 0, 1, \cdots. \tag{3.6}$$

Further, if $\{(B_k, M_k, N_k)\}$ are independent copies of (B, M, N) and if they are independent of (W, Z) then

$$\Big(B,\ \big(M - (d - ||W||^2)B\big),\ \big(N - BW\big)\Big) \stackrel{\mathrm{d}}{=} \frac{1}{m}\sum_{k=1}^{Z}\Big(B_k,\ \big(M_k - 2W\cdot N_k\big),\ N_k\Big). \tag{3.7}$$

Theorem 3.2. *For each $x = (x_1\cdots, x_d) \in \mathbf{R}^d$,*

$$\sqrt{T}\Big[\frac{\psi\big(\{y:\ y \le x\sqrt{T}\}, T\big)}{m^T} - B\Phi_d(x)\Big] \longrightarrow -\nabla\Phi_d(x)\cdot N \tag{3.8}$$

almost surely as well as in L_2-norm, as $T \to \infty$, where B is given in (1.1) and N is given in Theorem 3.1.

4. Proof of Theorem 2.1, Theorem 2.2 and Corollary 2.3

Proof of Theorem 2.1. To prove (2.1) we need only to verify (2.9).

Define, for $0 \le t \le T$ and $x \in \mathbf{Z}^d$,

$$f(x,T,t) = m^{T-t} \sum_{y \in \mathbf{Z}^d} \lambda(y,t) P^{T-t}(x-y).$$

According to Lemma 4.3, p. 67 in Révész (1994),

$$E\big[\lambda(x,T)|\mathcal{F}(t)\big] = f(x,T,t) \qquad 0 \le t \le T \quad x \in \mathbf{Z}^d. \tag{4.1}$$

We first follow the decomposition given in Révész (1994): Fix a number $\epsilon > 0$ (which is sufficiently small to satisfy all the needs in the later argument) and choose $t \sim T^\epsilon$. For all $x \in \mathbf{Z}^d$,

$$\begin{aligned} \frac{\lambda(x,T)}{m^T} - P^T(x)B &= \Big(\frac{\lambda(x,T)}{m^T} - \frac{f(x,T,t)}{m^T}\Big) \\ &+ \Big(\frac{f(x,T,t)}{m^T} - P^T(x)\frac{B(t)}{m^t}\Big) \\ &+ P^T(x)\Big(\frac{B(t)}{m^t} - B\Big). \end{aligned} \tag{4.2}$$

In the proof of Theorem 2.1, we assume that $t \equiv x_1 + \cdots x_d \mod(2)$. As shown in Révész (1994), the first and the third terms are negligible since (Lemma 4.8 in Révész (1994))

$$E\bigg(\sum_{y \in \mathbf{Z^d}} \Big(\frac{\lambda(y,T)}{m^T} - \frac{f(y,T,t)}{m^T}\Big)^2\bigg) \le C \cdot \frac{1}{m^t (T-t)^{d/2}} \tag{4.3}$$

for some constant $C > 0$, and since (Theorem 4.8 in Révész (1994))

$$E\Big(\frac{B(t)}{m^t} - B\Big)^2 = O(m^{-t}). \tag{4.4}$$

So we need to show that

$$T^{1+d/2}\Big(\frac{f(x,T,t)}{m^T} - P^T(x)\frac{B(t)}{m^t}\Big) \longrightarrow d\Big(\frac{d}{2\pi}\Big)^{d/2}\big(M + x \cdot N\big) \tag{4.5}$$

almost surely as well as in L_2-norm. Notice that

$$\frac{f(x,T,t)}{m^T} - P^T(x)\frac{B(t)}{m^t} = \frac{1}{m^t}\sum_{y \in \mathbf{Z}^d} \lambda(y,t)\big(P^{T-t}(x-y) - P^T(x)\big). \tag{4.6}$$

By a formula given in p.14, Lawler (1991),

$$\begin{aligned} P^{T-t}(x-y) &= 2(2\pi)^{-d}\int_A e^{-i(x-y)\cdot\lambda}\phi^{T-t}(\lambda)d\lambda \\ &= 2(2\pi)^{-d}\int_A \cos\big((x-y)\cdot\lambda\big)\phi^{T-t}(\lambda)d\lambda, \end{aligned}$$

$$P^T(x) = 2(2\pi)^{-d} \int_A \cos(x \cdot \lambda)\phi^T(\lambda)d\lambda,$$

where

$$\phi(\lambda) = \frac{1}{d}\sum_{j=1}^{d} \cos\lambda_j \qquad \lambda = (\lambda_1, \cdots, \lambda_d) \in \mathbf{R}^d$$

is the characteristic function of X and $A = [-\pi/2, \pi/2] \times [-\pi, \pi]^{d-1}$. By variable substitution

$$\begin{aligned}
&P^{T-t}(x-y) - P^T(x) \\
&= 2(2\pi)^{-d}(T-t)^{-d/2} \int_{\sqrt{T-t}A} \Big[\cos\Big(\frac{(x-y)\cdot\lambda}{\sqrt{T-t}}\Big) - \cos\Big(\frac{x\cdot\lambda}{\sqrt{T-t}}\Big)\phi\Big(\frac{\lambda}{\sqrt{T-t}}\Big)^t\Big] \\
&\qquad \times \phi\Big(\frac{\lambda}{\sqrt{T-t}}\Big)^{T-t} d\lambda
\end{aligned}$$

and, by Taylor's expansion

$$\begin{aligned}
&\cos\Big(\frac{(x-y)\cdot\lambda}{\sqrt{T-t}}\Big) - \cos\Big(\frac{x\cdot\lambda}{\sqrt{T-t}}\Big)\phi\Big(\frac{\lambda}{\sqrt{T-t}}\Big)^t \\
&= \frac{1}{2(T-t)}\Big[(x\cdot\lambda)^2 + td^{-1}||\lambda||^2 - \big((x-y)\cdot\lambda\big)^2 + o(1)\Big]
\end{aligned}$$

uniformly for all $||y|| \le t$ as $T \to \infty$. Notice that

$$\phi\Big(\frac{\lambda}{\sqrt{T-t}}\Big)^{T-t} \longrightarrow \exp\Big\{-\frac{||\lambda||^2}{2d}\Big\} \qquad (T \to \infty)$$

and that $|\phi(\lambda)| < 1$ for all $\lambda \in A \setminus \{0\}$. Hence the dominated convergence theorem applies (see, e.g., the proof of Theorem 1.2.1 in Lawler (1991)), which, combined with above observations, gives that

$$\begin{aligned}
&P^{T-t}(x-y) - P^T(x) \\
&= (2\pi)^{-d}T^{-1-d/2} \\
&\qquad \times \Big[\int \Big((x\cdot\lambda)^2 + td^{-1}||\lambda||^2 - \big((x-y)\cdot\lambda\big)^2\Big)\exp\Big\{-\frac{||\lambda||^2}{2d}\Big\}d\lambda + o(1)\Big] \\
&= (2\pi)^{-d/2}d^{1+d/2}T^{-1-d/2}\Big[t - ||y||^2 + 2x\cdot y + o(1)\Big] \qquad (T \to \infty)
\end{aligned} \tag{4.6}$$

uniformly on $||y|| \le t$. Since $\lambda(y,t) = 0$ for all $||y|| > t$, from (4.6) we have

$$\begin{aligned}
&T^{1+d/2}\Big(\frac{f(x,T,t)}{m^T} - P^T(x)\frac{B(t)}{m^t}\Big) \\
&= (2\pi)^{-d/2}d^{1+d/2}\Big[t\frac{B(t)}{m^t} - \frac{1}{m^t}\sum_{y\in\mathbf{Z}^d}||y||^2\lambda(y,t) \\
&\qquad + \frac{2}{m^t}x\cdot\sum_{y\in\mathbf{Z}^d} y\lambda(y,t) + o(1)\Big]
\end{aligned} \tag{4.7}$$

almost surely and in L_2-norm as well. Let

$$M_t = t\frac{B(t)}{m^t} - \frac{1}{m^t}\sum_{y\in\mathbf{Z}^d} ||y||^2\lambda(y,t) \quad \text{and} \quad N_t = \frac{1}{m^t}\sum_{y\in\mathbf{Z}^d} y\lambda(y,t) \qquad t = 0,1,\cdots.$$

We claim that $\{M_t\}_{t\geq 0}$ and $\{N_t\}_{t\geq 0}$ are martingales w.r.t. the filtration $\{\mathcal{F}(t)\}_{t\geq 0}$. Indeed,

$$\begin{aligned}
E\big[M_t|\mathcal{F}(t-1)\big] &= E\big[t\frac{B(t)}{m^t}|\mathcal{F}(t-1)\big] - \frac{1}{m^t}\sum_{y\in\mathbf{Z}^d} ||y||^2 E\big[\lambda(y,t)|\mathcal{F}(t-1)\big] \\
&= t\frac{B(t-1)}{m^{t-1}} - \frac{1}{m^{t-1}}\sum_{y\in\mathbf{Z}^d} ||y||^2 \sum_{z\in\mathbf{Z}^d}\lambda(z,t-1)P(y-z) \\
&= t\frac{B(t-1)}{m^{t-1}} - \frac{1}{m^{t-1}}\sum_{z\in\mathbf{Z}^d}\lambda(z,t-1)E||z+X||^2 \\
&= t\frac{B(t-1)}{m^{t-1}} - \frac{1}{m^{t-1}}\sum_{z\in\mathbf{Z}^d}\lambda(z,t-1)\big\{||z||^2+1\big\} = M_{t-1}.
\end{aligned} \tag{4.8}$$

The proof for $\{N_t\}_{t\geq 0}$ being a martingale is similar.

To apply the martingale convergence theorem to $\{M_t\}_{t\geq 0}$ and $\{N_t\}_{t\geq 0}$, we compute their second moments. Note that

$$\begin{aligned}
M_t &= \frac{1}{m^t}\sum_{y\in\mathbf{Z}^d}\big\{t-||y||^2\big\}\lambda(y,t) \\
&= \frac{1}{m^t}\sum_{y\in\mathbf{Z}^d}\big\{t-||y||^2\big\}\sum_{z\in\mathbf{Z}^d}\sum_{k=1}^{\lambda(z,t-1)} I_{y-z}\big(X(z,t-1,k)\big)Z(z,t-1,k) \\
&= \frac{1}{m^t}\sum_{z\in\mathbf{Z}^d}\sum_{k=1}^{\lambda(z,t-1)} Z(z,t-1,k)\sum_{y\in\mathbf{Z}^d}\big\{t-||y||^2\big\}I_{y-z}\big(X(z,t-1,k)\big) \\
&= \frac{1}{m^t}\sum_{z\in\mathbf{Z}^d}\sum_{k=1}^{\lambda(z,t-1)} Z(z,t-1,k)\big\{t-||z+X(z,t-1,k)||^2\big\}.
\end{aligned}$$

Thus for each $t\geq 1$,

$$\begin{aligned}
&M_t - M_{t-1} = M_t - E\big[M_t|\mathcal{F}(t-1)\big] \\
&= \frac{1}{m^t}\sum_{z\in\mathbf{Z}^d}\sum_{k=1}^{\lambda(z,t-1)}\Big\{Z(z,t-1,k)\big\{t-||z+X(z,t-1,k)||^2\big\} \\
&\qquad - E\Big(Z(z,t-1,k)\big\{t-||z+X(z,t-1,k)||^2\big\}\Big)\Big\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
E\big[M_t - M_{t-1}\big]^2 &= \frac{1}{m^{2t}} \sum_{z\in\mathbf{Z}^d} E\big(\lambda(z,t-1)\big)\cdot Var\Big\{Z\cdot\big[t-||z+X||^2\big]\Big\} \\
&= \frac{1}{m^{t+1}} \sum_{z\in\mathbf{Z}^d} P^{t-1}(z) Var\Big\{Z\cdot\big[t-||z+X||^2\big]\Big\} \\
&= \frac{1}{m^{t+1}}\cdot Var\Big\{Z\cdot\big[t-||S_t||^2\big]\Big\} = \frac{1}{m^{t+1}} EZ^2\cdot E\big[t-||S_t||^2\big]^2 \\
&= \frac{1}{m^{t+1}}(m^2+\sigma^2)\cdot 2(t^2-t)d^{-1}.
\end{aligned}$$

Therefore

$$\sum_{t=1}^{\infty} E\big[M_t - M_{t-1}\big]^2 = \frac{4(m^2+\sigma^2)}{d(m-1)^3} < \infty. \tag{4.9}$$

Similarly,

$$N_t = \frac{1}{m^t}\sum_{z\in\mathbf{Z}^d}\sum_{k=1}^{\lambda(z,t-1)} Z(z,t-1,k)\big(z+X(z,t-1,k)\big),$$

$$\begin{aligned}
N_t - N_{t-1} = \frac{1}{m^t}\sum_{z\in\mathbf{Z}^d}\sum_{k=1}^{\lambda(z,t-1)} \Big\{ & Z(z,t-1,k)\big(z+X(z,t-1,k)\big) \\
& - E\Big(Z(z,t-1,k)\big(z+X(z,t-1,k)\big)\Big)\Big\}.
\end{aligned}$$

Thus for each $t \geq 1$,

$$\begin{aligned}
&\mathbf{cov}(N_t - N_{t-1}, N_t - N_{t-1}) \\
&= \frac{1}{m^{t+1}}\sum_{z\in\mathbf{Z}^d}\sum_{z\in\mathbf{Z}^d} P^{t-1}(z)\mathbf{cov}\big(Z(z+X), Z(z+X)\big) \\
&= \frac{1}{m^{t+1}}\mathbf{cov}\big(ZS_t, ZS_t\big) = \frac{1}{m^{t+1}}(m^2+\sigma^2)d^{-1}t\mathbf{I}_d.
\end{aligned}$$

Consequently,

$$\sum_{t=1}^{\infty}\mathbf{cov}(N_t - N_{t-1}, N_t - N_{t-1}) = \frac{(m^2+\sigma^2)}{d(m-1)^2}\mathbf{I}_d. \tag{4.10}$$

By martingale convergence theorem (see, e.g., p.2, Hall-Heyde (1980)), $\{M_t\}_{t\geq 0}$ and $\{N_t\}_{t\geq 0}$ converge almost surely, as well as in L_2-norm, to a real valued random variable M and a $\mathbf{R}^d$-valued random variable N, respectively. Since $M_0 = 0$ and $N_0 = \mathbf{0}$, we have

$$EM = 0 \quad\text{and}\quad EN = \mathbf{0}.$$

By above computation,

$$EM^2 = \frac{4(m^2+\sigma^2)}{d(m-1)^3} \quad \text{and} \quad \mathbf{cov}(N,N) = \frac{(m^2+\sigma^2)}{d(m-1)^2}\mathbf{I}_d$$

In view of (4.7), (4.9), (4.10) and (4.11) we have (2.1)-(2.3), (2.5) and (2.6) in Theorem 2.1.

Replace $\{X(x,t,k)\}$ by $\{-X(x,t,k)\}$ and introduce the notations $\lambda'(x,t)$, M'_t, N'_t, M', N' for the replacements of $\lambda(x,t)$, M_t, N_t, M, N, resp., in our new particle system. By symmetry of migration we have

$$(B, M', N') \overset{\mathrm{d}}{=} (B, M, N). \tag{4.11}$$

On the other hand, $\lambda'(x,t) = \lambda(-x,t)$ for all $t \geq 0$ and $x \in \mathbf{Z}^d$. Hence we have $M'_t = M_t$ and $N'_t = -N_t$, which leads to $M' = M$ and $N' = -N$. Therefore, (2.4) follows from (4.11).

We now prove (2.7). Let $\lambda^*(x, T-1, k)$ be the population of the particles located at x at time T, who are descended from the original ancestor's k^{th} child, let

$$B_k(t) = \sum_{x\in\mathbf{Z}^d} \lambda^*(x, T-1, k)$$

and write

$$B_k = \lim_{t\to\infty} \frac{B_k(t)}{m^{t-1}}.$$

Then

$$\lambda(x,T) = \sum_{k=1}^{Z} \lambda^*(x, T-1, k) = \sum_z I_z(X) \sum_{k=1}^{Z} \lambda^*(x, T-1, k)$$

and, $\{B_k\}$ are i.i.d. random variables independent of (X, Z) and distributed as $\mathcal{L}(B)$. Notice that

$$\sum_z I_z(X) \sum_{k=1}^{Z} P^{T-1}(x-z) B_k = mB \sum_z I_z(X) P^{T-1}(x-z).$$

Similar to (4.6), for $||z|| = 1$,

$$\begin{aligned} P^{T-1}(x-z) - P^T(x) &= d\Big(\frac{d}{2\pi}\Big)^{d/2} T^{-1-d/2}\Big[1 - ||z||^2 + 2x\cdot z + o(1)\Big] \\ &= d\Big(\frac{d}{2\pi}\Big)^{d/2} T^{-1-d/2}\big(2x\cdot z + o(1)\big) \end{aligned}$$

as $T \to \infty$. Therefore,

$$
\begin{aligned}
&\sum_z I_z(X) \sum_{k=1}^{Z} \Big[\frac{\lambda^*(x, T-1, k)}{m^{T-1}} - B_k P^{T-1}(x-z) \Big] \\
&= m\Big(\frac{\lambda(x,T)}{m^T} - BP^T(x) \Big) - mBd \Big(\frac{d}{2\pi} \Big)^{d/2} T^{-1-d/2} \sum_z I_z(X) \big(2x \cdot z + o(1) \big) \\
&= m\Big(\frac{\lambda(x,T)}{m^T} - BP^T(x) \Big) - mBd \Big(\frac{d}{2\pi} \Big)^{d/2} T^{-1-d/2} \big(2x \cdot X + o(1) \big) \quad \text{a.s.}
\end{aligned}
$$

Applying (2.9) to above relation we have

$$
M + 2x \cdot (N - BX) = \frac{1}{m} \sum_{k=1}^{Z} \Big\{ (M_k - 2X \cdot N_k) + 2x \cdot N_k \Big\} \qquad \text{a.s.,}
$$

where $\{(M_k, N_k)\}$ are independent copies of (M, N) and they are independent of (X, Z). Since $x \in \mathbf{Z}^d$ is arbitrary and since

$$
B = \frac{1}{m} \sum_{k=1}^{Z} B_k,
$$

we have (2.7). ■

Proof of Theorem 2.2. Consider the following decomposition:

$$
\begin{aligned}
&\frac{1}{m^T} \sum_{y \le x\sqrt{T}} \lambda(y, T) - BP\{S_T \le x\sqrt{T}\} \\
&= \frac{1}{m^T} \sum_{y \le x\sqrt{T}} \big(\lambda(y,T) - f(y,T,t) \big) \\
&+ \sum_{y \le x\sqrt{T}} \frac{f(y,T,t)}{m^T} - P\{S_T \le x\sqrt{T}\} \frac{B(t)}{m^t} \\
&+ P\{S_T \le x\sqrt{T}\} \Big(\frac{B(t)}{m^t} - B \Big).
\end{aligned}
$$

Again, we let $t \sim T^\epsilon$ for some sufficiently small $\epsilon > 0$. In view of (4.4), the third term in above decomposition is negligible. Since $\lambda(y,T) = 0$ for all $||y|| > T$

$$
\Big(\sum_{y \le x\sqrt{T}} \big(\lambda(y,T) - f(y,T,t) \big) \Big)^2 \le KT^{d/2} \sum_{y \in \mathbf{Z}^d} \big(\lambda(y,T) - f(y,T,t) \big)^2,
$$

where $K > 0$ is a constant. By (4.3), the first term is also negligible.

Therefore, we only need to deal with the second term, i.e, to prove that

$$\sqrt{T}\Big[\sum_{y\le x\sqrt{T}}\frac{f(y,T,t)}{m^T}-P\big\{S_T\le x\sqrt{T}\big\}\frac{B(t)}{m^t}\Big]\longrightarrow -\nabla\Phi_d(\sqrt{d}x)\cdot N \tag{4.12}$$

almost surely as well as in L_2-norm.

Notice that

$$\begin{aligned}&\frac{1}{m^T}\sum_{y\le x\sqrt{T}}f(y,T,t)-P\big\{S_T\le x\sqrt{T}\big\}\frac{B(t)}{m^t}\\&=\frac{1}{m^t}\sum_{y\in\mathbf{Z}^d}\Big[P\big\{S_{T-t}\le x\sqrt{T}-y\big\}-P\big\{S_T\le x\sqrt{T}\big\}\Big]\lambda(y,t).\end{aligned}\tag{4.13}$$

Write

$$\begin{aligned}&P\big\{S_{T-t}\le x\sqrt{T}-y\big\}-P\big\{S_T\le x\sqrt{T}\big\}\\&=\Big(P\big\{S_T\le x\sqrt{T}-y\big\}-P\big\{S_T\le x\sqrt{T}\big\}\Big)\\&+\Big(P\big\{S_{T-t}\le x\sqrt{T}-y\big\}-P\big\{S_T\le x\sqrt{T}-y\big\}\Big)\\&=(I)+(II)\qquad\text{(say)}.\end{aligned}\tag{4.14}$$

Uniformly on $||y||\le t$,

$$\begin{aligned}(I)=-\big(1+o(1)\big)\sum_{j=1}^d\operatorname{sgn}(y_j)P\Big\{&x_j\sqrt{T}-|y_j|<S_T^{(j)}\le x_j\sqrt{T},\\&S_T^{(k)}\le x_k\sqrt{T},\ k\ne j,\ 1\le k\le d\Big\}\end{aligned}\tag{4.15}$$

as $T\to\infty$, where $S_T^{(1)},\cdots,S_T^{(d)}$ are the components of S_T. We claim that for each $1\le j\le d$, uniformly on $||y||\le t$,

$$\begin{aligned}&P\Big\{x_j\sqrt{T}-|y_j|<S_T^{(j)}\le x_j\sqrt{T},\quad S_T^{(k)}\le x_k\sqrt{T},\ k\ne j,\ 1\le k\le d\Big\}\\&=\begin{cases}\big(1+o(1)\big)T^{-1/2}\dfrac{d}{dx}\big(\Phi(x)\big)\Big[|y|+\big(1-(-1)^{|y|}\big)(-1)^{T+[x\sqrt{T}]}\Big] & \text{as } d=1,\\[2ex]\big(1+o(1)\big)T^{-1/2}|y_j|\dfrac{\partial}{\partial x_j}\Big(\Phi_d(\sqrt{d}x)\Big) & \text{as } d\ge 2.\end{cases}\end{aligned}\tag{4.16}$$

We only prove (4.16) in the case $d\ge 2$, as the proof for the case $d=1$ is similar but mush simpler. Without loss of generality, we only consider the case $j=1$. By a combinatorial argument we can see that for any measurable $A_1,\cdots,A_d\subset\mathbf{R}$,

$$\begin{aligned}&P\big\{S_T^{(1)}\in A_1,\cdots,S_T^{(d)}\in A_d\big\}\\&=\frac{1}{d^T}\sum_{k_1+\cdots+k_d=T}\frac{T!}{k_1!\cdots k_d!}\prod_{j=1}^d P\big\{\bar{S}_{k_j}\in A_j\big\},\end{aligned}\tag{4.17}$$

where $\bar{S}_t$ is an 1-dimensional symmetric random walk. In particular,

$$
\begin{aligned}
&P\Big\{x_1\sqrt{T}-|y_1|<S_T^{(1)}\le x_1\sqrt{T},\ S_T^{(k)}\le x_k\sqrt{T},\ 2\le k\le d\Big\}\\
&=\frac{1}{d^T}\sum_{k_1+\cdots+k_d=T}\frac{T!}{k_1!\cdots k_d!}\bigg(\sum_{x_1\sqrt{T}-|y_1|<z\le x_1\sqrt{T}}P\big\{\bar{S}_{k_1}=z\big\}\bigg)\prod_{j=2}^d P\big\{\bar{S}_{k_j}\le x_j\sqrt{T}\big\}.
\end{aligned}
$$

Notice that $P\big\{\bar{S}_{k_1}=z\big\}=0$ when $k_1\not\equiv z$ mod (2), and

$$
P\big\{\bar{S}_{k_1}=z\big\}\sim 2T^{-1/2}\sqrt{\frac{d}{2\pi}}\exp\Big\{-\frac{dx_1^2}{2}\Big\}
$$

when $k_1\equiv z$ mod (2) and $k_1\sim d^{-1}T$ as $T\to\infty$. On the other hand,

$$
\begin{aligned}
&\#\big\{z\in\mathbf{Z}:\quad x_1\sqrt{T}-|y_1|<z\le x_1\sqrt{T}\ \text{and}\ z\equiv k_1\ mod(2)\big\}\\
&=\frac{1}{2}\Big[|y_1|+\big(1-(-1)^{|y_1|}\big)(-1)^{k_1+[x_1\sqrt{T}]}\Big].
\end{aligned}
$$

Consider the Cramér's large deviation (cf., Theorem 2.2.30 of Dembo-Zeitouni (1992)) which gives in our case

$$
\frac{1}{d^T}\sum_{(k_1,\cdots,k_d)\in A_T(\delta)}\frac{T!}{k_1!\cdots k_d!}=O\Big(e^{-\alpha T}\Big)\qquad(T\to\infty)\tag{4.18}
$$

for any $\delta>0$, where $\alpha=\alpha(\delta)>0$ is a constant and

$$
\begin{aligned}
A_T(\delta)=\big\{&(k_1,\cdots,k_d)\in\mathbf{Z}^d;\quad k_1+\cdots+k_d=T,\ k_j\ge 0\ (j=1,\cdots,d)\\
&\text{and}\ |k_j-d^{-1}T|\ge\delta T\ \text{for some}\ 1\le j\le d\big\}
\end{aligned}
$$

(We only need that the left hand side of (4.18) tend to zero here, which is referred as the law of large numbers. We give (4.18) for the needs of the later development). Therefore,

$$
\begin{aligned}
&P\Big\{x_1\sqrt{T}-|y_1|<S_T^{(1)}\le x_1\sqrt{T}+|y_1|,\ S_T^{(k)}\le x_k\sqrt{T},\ 2\le k\le d\Big\}\\
&=\big(1+o(1)\big)\frac{1}{d^T}\sum_{k_1+\cdots+k_d=T}\frac{T!}{k_1!\cdots k_d!}\Big[|y_1|+\big(1-(-1)^{|y_1|}\big)(-1)^{k_1+[x_k\sqrt{T}]}\Big]\\
&\qquad\qquad\times T^{-1/2}\sqrt{\frac{d}{2\pi}}\exp\Big\{-\frac{dx_1^2}{2}\Big\}\prod_{j=2}^d\Phi\big(\sqrt{d}x_j\big)\\
&=\big(1+o(1)\big)T^{-1/2}\frac{\partial}{\partial x_1}\Big(\Phi_d(\sqrt{d}x)\Big)\Big[|y_1|+\big(1-(-1)^{|y_1|}\big)(-1)^{[x_1\sqrt{T}]}\Big(1-\frac{2}{d}\Big)^T\Big]\\
&=\big(1+o(1)\big)T^{-1/2}\frac{\partial}{\partial x_1}\Big(\Phi_d(\sqrt{d}x)\Big)|y_1|\qquad(T\to\infty).
\end{aligned}
$$

By (4.15) and (4.16),

(4.19)
$$(I)=\begin{cases} -T^{-1/2}\dfrac{d}{dx}\big(\Phi(x)\big)\Big[y+\big(1-(-1)^{|y|}\big)\mathrm{sgn}(y)(-1)^{T+[x\sqrt{T}]}+o(1)\Big] & \text{as } d=1,\\ -T^{-1/2}\Big[\nabla\Phi_d(\sqrt{d}x)\cdot y+o(1)\Big] & \text{as } d\ge 2\end{cases}$$

holds uniformly on $||y||\le t$ as $T\to\infty$.

Note that when $d\ge 2$,

$$\begin{aligned}(II)&=\sum_{z\in\mathbf{Z}^d}P^t(z)\Big[P\big\{S_{T-t}\le x\sqrt{T}-y\big\}-P\big\{S_{T-t}\le x\sqrt{T}-y-z\big\}\Big]\\ &=-\sum_{z\in\mathbf{Z}^d}P^t(z)T^{-1/2}\Big[\nabla\Phi_d(\sqrt{d}x)\cdot z+o(1)\Big] \qquad (4.20)\\ &=-T^{-1/2}\Big[E\big(\nabla\Phi_d(\sqrt{d}x)\cdot S_t\big)+o(1)\Big]=o\big(T^{-1/2}\big)\quad (T\to\infty)\end{aligned}$$

holds uniformly on $||y||\le t$, where the second equality follows from an obvious modification of (4.19). Similarly, (4.20) is also true when $d=1$.

Combining (4.13), (4.14), (4.19), (4.20) we can see that as $d\ge 2$,

$$\begin{aligned}&\frac{1}{m^T}\sum_{y\le x\sqrt{T}}f(y,T,t)-P\big\{S_T\le x\sqrt{T}\big\}\frac{B(t)}{m^t}\\ &=-T^{-1/2}\Big[\nabla\Phi_d(\sqrt{d}x)\cdot\Big(\frac{1}{m^t}\sum_{y\in\mathbf{Z}^d}y\lambda(y,t)\Big)+o(1)\Big]\end{aligned}$$

almost surely as well as in L_2-norm. This also holds in the case $d=1$ as

$$\begin{aligned}&\Big|\frac{1}{m^T}\sum_{y\in\mathbf{Z}}\big(1-(-1)^{|y|}\big)\mathrm{sgn}(y)\lambda(y,t)\Big|\\ &=\Big|\sum_{y\in\mathbf{Z}}\big(1-(-1)^{|y|}\big)\mathrm{sgn}(y)\Big[\frac{\lambda(y,t)}{m^t}-P^t(y)B\Big]\Big|\\ &\le\sum_{y\in\mathbf{Z}}\Big|\frac{\lambda(y,t)}{m^t}-P^t(y)B\Big|\longrightarrow 0\qquad (T\to\infty)\end{aligned}$$

where the last step follows from Theorem 4.3 and Theorem 4.5 in Révész (1994).

So (4.12) follows from (2.6) and martingale convergence theorem. ∎

Proof of Corollary 2.3. By Theorem 2.2, it is enough to prove the following multivariate version of Edgeworth expansion:

$$P\{S_T\le x\sqrt{T}\}=\Phi_d(\sqrt{d}x)+T^{-1/2}\nabla\Phi_d(\sqrt{d}x)\cdot\mathbf{F}(\sqrt{T}x)+o\big(T^{-1/2}\big). \qquad (4.21)$$

When $d = 1$, by Theorem 6, p.171 of Petrov (1975) we have

$$
\begin{aligned}
P\{S_T \le x\sqrt{T}\} &= \Phi(x) + \frac{1}{\sqrt{T}}\Phi'(x)\sum_{l=1}^{\infty}\frac{\sin\big(2l\pi\sqrt{T}x\big)}{\pi l} + +o\big(T^{-1/2}\big) \\
&= \Phi(x) + T^{-1/2}\Phi'(x)f(\sqrt{T}x) + o\big(T^{-1/2}\big),
\end{aligned} \tag{4.22}
$$

where the last step follows from Fourier expansion.

Consider the case when $d > 1$. From (4.17)

$$
P\big\{S_T \le x\sqrt{T}\big\} = \frac{1}{d^T}\sum_{k_1+\cdots+k_d=T}\frac{T!}{k_1!\cdots k_d!}\prod_{j=1}^{d}P\big\{\bar{S}_{k_j} \le x_j\sqrt{T}\big\}. \tag{4.23}
$$

By a treatment similar to the one used in the proof of (4.16), the expansion (4.21) follows from (4.18), (4.22) and (4.23). ■

5. Proof of Theorem 3.1 and Theorem 3.2

Let $P_t(x, A)$ be the transition probability of $\{W(t)\}$. Then

$$
P_t(x, A) = \Big(\frac{1}{2\pi t}\Big)^{d/2}\int_A \exp\Big\{-\frac{||y-x||^2}{2t}\Big\}dy \qquad t > 0.
$$

Define, for each $t \le T$,

$$
F(A, T, t) = m^{T-t}\sum_{x\in\mathbf{R}^d}P_{T-t}(x, A)\lambda(x, t) = m^{T-t}\int P_{T-t}(x, A)\psi(dx, t).
$$

By Lemma 6.3 in Révész (1994), for each $0 \le t < T$,

$$
E\big[\psi(A, T)|\mathcal{F}(t)\big] = F(A, T, t). \tag{5.1}
$$

The proof here is similar to the one given in section 4, except that the range of branching Wiener process is no longer bounded in probability. We shall overcome such a difficulty by truncation. From this motive we need the following lemma.

Lemma 5.1. *There exists a constant $C > 0$ depending only on m and σ^2 such that*

$$
E\Big[\int f(x)\Psi(dx, t)\Big]^2 \le Cm^{2t}Ef^2\big(W(t)\big) \tag{5.2}
$$

for every function f on $\mathbf{R}^d$, and t.

Proof. By (5.1),

$$
E\Big[\int f(x)\Psi(dx, t)\Big] = \int f(x)F(dx, t, 0) = m^tEf\big(W(t)\big). \tag{5.3}
$$

Notice that

$$\int f(x)\Psi(dx,t) - E\Big[\int f(x)\Psi(dx,t)\Big]$$
$$= \sum_{s=1}^{t}\Big[\int f(x)F(dx,t,s) - \int f(x)F(dx,t,s-1)\Big],$$

$$\int f(x)F(dx,t,s) = m^{t-s}\sum_{x\in\mathbf{R}^d} P_{t-s}f(x)\lambda(x,t)$$
$$= m^{t-s}\sum_{x\in\mathbf{R}^d} P_{t-s}f(x)\sum_{y\in\mathbf{R}^d}\sum_{k=1}^{\lambda(y,t-1)} I_x\big(y+W(y,t-1,k)\big)Z(y,t-1,k)$$
$$= m^{t-s}\sum_{y\in\mathbf{R}^d}\sum_{k=1}^{\lambda(y,s-1)} P_{t-s}f\big(y+W(y,s-1,k)\big)Z(y,s-1,k).$$

In view of (5.1) we have

$$\int f(x)F(dx,t,s) - \int f(x)F(dx,t,s-1)$$
$$= m^{t-s}\sum_{y\in\mathbf{R}^d}\sum_{k=1}^{\lambda(y,s-1)}\Big[P_{t-s}f\big(y+W(y,s-1,k)\big)Z(y,s-1,k)$$
$$- E\Big(P_{t-s}f\big(y+W(y,s-1,k)\big)Z(y,s-1,k)\Big)\Big].$$

Hence,

$$E\Big[\int f(x)F(dx,t,s) - \int f(x)F(dx,t,s-1)\Big]^2$$
$$= m^{2t-2s}\sum_{y\in\mathbf{R}^d} E\big(\lambda(y,s-1)\big)Var\Big(P_{t-s}f(y+W)Z\Big)$$
$$= m^{2t-s-1}\int P_{s-1}(0,dy)Var\Big(P_{t-s}f(y+W,A)Z\Big)$$
$$= m^{2t-s-1}Var\Big(P_{t-s}f\big(W(s)\big)Z\Big)$$
$$\le m^{2t-s-1}(m^2+\sigma^2)E\Big(P_{t-s}f\big(W(s)\big)\Big)^2$$
$$\le m^{2t-s-1}(m^2+\sigma^2)E\Big(P_{t-s}f^2\big(W(s)\big)\Big)$$
$$= m^{2t-s-1}(m^2+\sigma^2)Ef^2\big(W(t)\big).$$

Therefore, by orthogonality ((5.1)) we have

$$
\begin{aligned}
Var\Big[\int f(x)\Psi(dx,t)\Big] &= \sum_{s=1}^{t} m^{2t-s-1}(m^2+\sigma^2)Ef^2\big(W(t)\big) \\
&\le m^{2t}\cdot\frac{m^2+\sigma^2}{m(m-1)}Ef^2\big(W(t)\big).
\end{aligned}
\tag{5.4}
$$

Finally, the desired conclusion follows from (5.3) and (5.4). ■

To prove Theorem 3.1 and 3.2, we now follow Révész' decomposition again

$$
\begin{aligned}
&\frac{\psi(A,T)}{m^T} - B\Big(\frac{1}{2\pi T}\Big)^{d/2}\int_A \exp\Big\{-\frac{||x||^2}{2T}\Big\}dx \\
&= \Big(\frac{\psi(A,T)}{m^T} - \frac{F(A,T,t)}{m^T}\Big) \\
&+ \Big(\frac{F(A,T,t)}{m^T} - P_T(0,A)\frac{B(t)}{m^t}\Big) + P_T(0,A)\Big(\frac{B(t)}{m^t} - B\Big),
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\psi\big(\{y;\ y\le x\sqrt{T}\},T\big)}{m^T} - B\Phi_d(x) \\
&= \Big(\frac{\psi\big(\{y;\ y\le x\sqrt{T}\},T\big)}{m^T} - \frac{F\big(\{y;\ y\le x\sqrt{T}\},T,t\big)}{m^T}\Big) \\
&+ \Big(\frac{F\big(\{y;\ y\le x\sqrt{T}\},T,t\big)}{m^T} - \Phi_d(x)\frac{B(t)}{m^t}\Big) + \Phi_d(x)\Big(\frac{B(t)}{m^t} - B\Big),
\end{aligned}
$$

where we let $t = T^\epsilon$ for a small constant $\epsilon > 0$. By (4.4) and the inequality given in Lemma 6.11 of Révész (1994), one can see that only the second term in each of these two decompositions contributes to the limit behaviors given in our theorems. In other words, (3.1) and (3.8) are equivalent to

$$
T^{1+d/2}\Big(\frac{F(A,T,t)}{m^T} - P_T(0,A)\frac{B(t)}{m^t}\Big) \longrightarrow |A|\Big(\frac{1}{2\pi}\Big)^{d/2}\big(\frac{1}{2}M + \bar{x}_A\cdot N\big)
\tag{5.5}
$$

and

$$
\sqrt{T}\Big(\frac{F\big(\{y;\ y\le x\sqrt{T}\},T,t\big)}{m^T} - \Phi_d(x)\frac{B(t)}{m^t}\Big) \longrightarrow -\nabla\Phi_d(x)\cdot N,
\tag{5.6}
$$

respectively.

We first prove (5.5). Notice that

$$
\begin{aligned}
&\frac{F(A,T,t)}{m^T} - P_T(0,A)\frac{B(t)}{m^t} \\
&= \frac{1}{m^t}\int\big[P_{T-t}(x,A) - P_T(0,A)\big]\psi(dx,t) \\
&= \frac{1}{m^t}\int_{|x|\le t}\big[P_{T-t}(x,A) - P_T(0,A)\big]\psi(dx,t) \\
&+ \frac{1}{m^t}\int_{|x|>t}\big[P_{T-t}(x,A) - P_T(0,A)\big]\psi(dx,t)
\end{aligned}
$$

and that, by Lemma 5.1

$$E\Big[\frac{1}{m^t}\int_{|x|>t}\big[P_{T-t}(x,A)-P_T(0,A)\big]\psi(dx,t)\Big]^2 = O\Big(P\{|W(t)|>t\}\Big) = O\Big(e^{-ct}\Big).$$

So (5.5) is equivalent to

$$\text{(5.7)}\quad T^{1+d/2}\Big[\frac{1}{m^t}\int_{|x|\le t}\big[P_{T-t}(x,A)-P_T(0,A)\big]\psi(dx,t)\Big] \longrightarrow |A|\Big(\frac{1}{2\pi}\Big)^{d/2}\big(\frac{1}{2}M+\bar{x}_A\cdot N\big).$$

On the other hand,

$$\begin{aligned}&P_{T-t}(x,A)-P_T(0,A)\\ &=\frac{1}{2}\Big(\frac{1}{2\pi}\Big)^{d/2}|A|T^{-1-d/2}\Big[dt-||x||^2+\frac{2}{|A|}\int_A(x\cdot y)dy+o(1)\Big],\end{aligned}$$

where the error term $o(1)$ tends to 0 uniformly over $|x|\le t$ as $T\to\infty$. Hence

$$\begin{aligned}&T^{1+d/2}\frac{1}{m^t}\int_{|x|\le t}\big[P_{T-t}(x,A)-P_T(0,A)\big]\psi(dx,t)\\ &=\frac{1}{2}\Big(\frac{1}{2\pi}\Big)^{d/2}\frac{|A|}{m^t}\int_{||x||\le t}\Big[dt-|x|^2+\frac{2}{|A|}\int_A(x\cdot y)dy\Big]\psi(dx,t)+o(1)\\ &=\frac{1}{2}\Big(\frac{1}{2\pi}\Big)^{d/2}|A|\Big[dt\frac{B(t)}{m^t}-\frac{1}{m^t}\int|x|^2\psi(dx,t)+2\bar{x}_A\frac{1}{m^t}\int x\psi(dx,t)\Big]+o(1)\end{aligned}$$

almost surely as well as in L_2-norm, where the last step follows from a truncation estimate via Lemma 5.1.

Similarly as in section 4, we can show that

$$M_t\equiv dt\frac{B(t)}{m^t}-\frac{1}{m^t}\int||x||^2\psi(dx,t)\quad\text{and}\quad N_t\equiv\frac{1}{m^t}\int x\psi(dx,t)$$

are two martingales with

$$E\big[M_t-M_{t-1}\big]^2=\frac{2}{m^{t+1}}(m^2+\sigma^2)dt^2$$

$$\mathbf{cov}\big(N_t-N_{t-1},N_t-N_{t-1}\big)=\frac{1}{m^{t+1}}(m^2+\sigma^2)\mathbf{I}_d$$

for all $t\ge 1$. By martingale convergence theorem $\{M_t\}$ and $\{N_t\}$ converge, almost surely as well as in L_2-norm, to some random variables M and N, respectively. Further,

$$EM^2=\sum_{t=1}^{\infty}E\big[M_t-M_{t-1}\big]^2=\frac{2d(m^2+\sigma^2)(m+1)}{(m-1)^3},$$

$$\mathbf{cov}\big(N,N\big)=\sum_{t=1}^{\infty}\mathbf{cov}\big(N_t-N_{t-1},N_t-N_{t-1}\big)=\frac{(m^2+\sigma^2)}{(m-1)^2}\mathbf{I}_d.$$

Hence we have (5.7) (and therefore (3.1)-(3.3), (3.5), (3.6)). We omit the proofs of (3.4) and (3.7), as they are analogous to that of (2.4)and (2.7), respectively.

We now come to the proof of (5.6). Note that for all $|y|\le t$, uniformly we have

$$\Phi_d\Big(\sqrt{\frac{T}{T-t}}x-\frac{y}{\sqrt{T-t}}\Big)-\Phi_d(x)=-T^{-1/2}\Big[\nabla\Phi_d(x)\cdot y+o(1)\Big].$$

Hence, by truncation (Lemma 5.1) we have

$$\begin{aligned}&\sqrt{T}\Big(\frac{F\big(\{y;\ y\le x\sqrt{T}\},T,t\big)}{m^T}-\Phi_d(x)\frac{B(t)}{m^t}\Big)\\&=\sqrt{T}\frac{1}{m^t}\int\big[P\big\{W(T-t)\le x\sqrt{T}-y\big\}-\Phi_d(x)\big]\psi(dy,t)\\&=-\nabla\Phi_d(x)\cdot\Big(\frac{1}{m^t}\int y\psi(dy,t)\Big)+o(1)\end{aligned}$$

almost surely as well as in L_2-norm. So (5.6) follows from (3.6) and martingale convergence theorem. ■

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