Homework # 2

Chapter 18

1, 6, 9, 10, 14, 15

1. Applying Ito-formula to $f(x) = e^x$

$$e^{B_t} - 1 = \int_0^t e^{B_s} dB_s + \frac{1}{2}\int_0^t e^{B_s} ds$$

Or,

$$\int_0^t e^{B_s} dB_s = e^{B_t} - 1 - \frac{1}{2}\int_0^t e^{B_s} ds$$

Applying Ito-formula to $f(x) = e^{x^2}$

$$e^{B_t^2} - 1 = 2\int_0^t B_s e^{B_s^2} dB_s + \int_0^t (1 + 2B_s^2) e^{B_s^2} ds$$

Or

$$\int_0^t B_s e^{B_s^2} dB_s = \frac{1}{2}\left\{ e^{B_t^2} - 1 - \int_0^t (1 + 2B_s^2) e^{B_s^2} ds \right\}$$

A sideremark: The Ito-integral in the second case is extented Ito-integral for $t \geq 1/2$..

6. (a). Nothing more than a integration by parts formula.

(b). Let $f(x, y) = xy$ and consider the 2-dimensional Brownian motion $B_t = (b_t, \beta_t)$. By Ito-formula

$$f(B_t) = \int_0^t \nabla f(B_s) \cdot dB_s + \frac{1}{2}\int_0^t \Delta f(B_s) ds$$

Notice that $\Delta f(x, y) = 0$ and $\nabla f(x, y) = \langle y, x \rangle$. So we have

$$b_t \beta_t = \int_0^t \beta_s db_s + \int_0^t b_s d\beta_s$$

When b_t and β_t are not independent,

$$f(B_t) = \int_0^t \nabla f(B_s) \cdot dB_s + \frac{1}{2}\int_0^t \Delta f(B_s) ds + \int_0^t \frac{\partial^2 f}{\partial x \partial y}(B_s) d\langle b, \beta \rangle_s$$

$$= \int_0^t \beta_s db_s + \int_0^t b_s d\beta_s + \langle b, \beta \rangle_t = \int_0^t \beta_s db_s + \int_0^t b_s d\beta_s + E(b_1 \beta_1) t$$

where the last step follows from

$$\langle b, \beta \rangle_t = \frac{1}{4}\left\{ \langle b + \beta \rangle_t - \langle b - \beta \rangle_t \right\} = \frac{1}{4}\left\{ E(b_1 + \beta_1)^2 t - E(b_1 - \beta_1)^2 t \right\} = tEb_1\beta_1$$

So

$$\int_0^t b_s d\beta_s = b_t\beta_t - \int_0^t \beta_s db_s - E(b_1\beta_1)t$$

A side remark. In the extreme case when $b_t = \beta_t$. The above identity becomes:

$$\int_0^t b_s db_s = b_t^2 - \int_0^t b_s db_s - t \quad \text{or} \quad b_t^2 = 2\int_0^t b_s db_s + t,$$

a formula we derived before.

9. Applying Ito-formula (Theorem 18.11, p.310) to the function $f(t,x) = e^{t/2}\cos x$

$$X_t = f(t,B_t) = f(0,0) + \int_0^t \frac{\partial f}{\partial x}(s,B_s)dB_s + \int_0^t \Big\{\frac{\partial f}{\partial t}(s,B_s) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s,B_s)\Big\}ds$$
$$= 1 - \int_0^t e^{s/2}\sin B_s dB_s$$

where the last step follows from

$$\frac{\partial f}{\partial t}(t,x) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,x) = 0$$

Applying Ito-formula (Corollary 17.10, p.261) to the function

$$f(t,x) = (x+t)\exp\{-x - t/2\}$$

we have

$$X_t = f(t,B_t) = f(0,0) + \int_0^t \frac{\partial f}{\partial x}(s,B_s)dB_s + \int_0^t \Big\{\frac{\partial f}{\partial t}(s,B_s) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s,B_s)\Big\}ds$$
$$= \int_0^t (1 - B_s - s)\exp\{-B_s - s/2\}dB_s$$

Remark. Both examples shows that $f(t,B_t)$ is a martingale.

10. (a). By the fact that

$$\Big|\frac{b_s}{r_s}\Big| \le 1 \quad \text{and} \quad \Big|\frac{\beta_s}{r_s}\Big| \le 1$$

we have

$$E\int_0^t \Big|\frac{b_s}{r_s}\Big|^2 ds \le t < \infty \quad \text{and} \quad E\int_0^t \Big|\frac{\beta_s}{r_s}\Big|^2 ds \le t < \infty$$

Therefore the Ito integrals

$$\int_0^t \frac{b_s}{r_s}db_s \quad \text{and} \quad \int_0^t \frac{\beta_s}{r_s}d\beta_s$$

are well-defined.

(b). Notice that under the notation $B(t) = (b_t, \beta_t)$,

$$\int_0^t \frac{b_s}{r_s} db_s nt_0^t \frac{b_s}{r_s} db_s + \int_0^t \frac{\beta_s}{r_s} d\beta_s = \int_0^t \frac{B_s}{\|B_s\|} \cdot dB_s$$

is a square integrable and continous martingale with the quadratic variation

$$\left\langle \int_0^{\cdot} \frac{B_s}{\|B_s\|} \cdot dB_s \right\rangle_t = \int_0^t \left\| \frac{B_s}{\|B_s\|} \right\|^2 ds = t$$

Thus, the martingale

$$\int_0^t \frac{B_s}{\|B_s\|} \cdot dB_s \qquad (t \geq 0)$$

is a 1-dimensional Brownian motion.

14. β_t is a square integrable and continuous martingale. To make it a 1-dimensional Brownian motion, all we need is to show $\langle \beta \rangle_t = t$. Indeed,

$$\langle \beta \rangle_t = \int_0^t |\mathrm{sgn}(B_s)|^2 ds = t \qquad a.s.$$

15. Set

$$M_t = \int_0^t g(r) dB_r$$

By Ito-formula

$$\Phi(M_t) = \Phi(0) + \int_0^t \Phi'(M_s) dM_s + \frac{1}{2} \int_0^t \Phi"(M_s) d\langle M \rangle_s$$
$$= \int_0^t \Phi'(M_s) g(B_s) dB_s + \frac{1}{2} \int_0^t \Phi"(M_s) g^2(B_s) ds$$

Hence

$$E\Big[\Big(\int_0^t f(B_s) dB_s\Big) \Phi\Big(\int_0^t g(r) dB_r\Big)\Big]$$
$$= E\Big[\Big(\int_0^t f(B_s) dB_s\Big)\Big(\int_0^t \Phi'(M_s) g(B_s) dB_s\Big)\Big]$$
$$+ \frac{1}{2} E\Big[\Big(\int_0^t f(B_s) dB_s\Big)\Big(\int_0^t \Phi"(M_s) g^2(B_s) ds\Big)\Big]$$

For the first term,

$$E\Big[\Big(\int_0^t f(B_s) dB_s\Big)\Big(\int_0^t \Phi'(M_s) g(B_s) dB_s\Big)\Big] = E \int_0^t f(B_s) \Phi'(M_s) g(B_s) ds$$
$$= E \int_0^t f(B_s) g(B_s) \Phi'\Big(\int_0^s g(r) dB_r\Big) ds$$

As for the second term, it is better to keep it in the current form than writting it as touble integral due to measurability issue.