H.W.S.

Ex 1.1.
Solution: Let \( \pi = (\pi_1, \pi_2, \pi_3) \) we know \( \pi_1 + \pi_2 + \pi_3 = 1 \)
and \( \pi (I - P) = 0 \) \( \Rightarrow (\pi_1, \pi_2, \pi_3) \begin{pmatrix} 0.3 & -0.2 & -0.1 \\ 0 & 0.4 & -0.4 \\ -0.5 & 0 & 0.5 \end{pmatrix} = 0 \)
\( \Rightarrow \begin{cases} 0.3\pi_1 - 0.5\pi_3 = 0 \\ -0.2\pi_1 + 0.4\pi_2 = 0 \\ -0.1\pi_1 - 0.4\pi_2 + 0.5\pi_3 = 0 \end{cases} \)
\( \Rightarrow \pi_1 = \frac{10}{21} \pi_2 = \frac{5}{21} \pi_3 = \frac{6}{21} \left(= \frac{2}{7}\right) \)
therefore, the limiting distribution is \( \left(\frac{10}{21}, \frac{5}{21}, \frac{6}{21}\right) \)

Prob 1.5. Sol: Based on the information, we can set up the transmission matrix:

\[
P = \begin{pmatrix}
A & B & C & D \\
0 & \frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\
0 & 1 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0
\end{pmatrix}
\]

therefore say \( \pi = (\pi_A, \pi_B, \pi_C, \pi_D) \), where \( \pi_A + \pi_B + \pi_C + \pi_D = 1 \)
and
\[ \pi (I - P) = 0 \]
\[ \Rightarrow \begin{cases} \pi_A - \frac{1}{3}\pi_B - \frac{1}{2}\pi_D = 0 \\ -\frac{1}{2}\pi_A + \pi_B - \frac{1}{2}\pi_D = 0 \\ -\frac{1}{3}\pi_B + \pi_C = 0 \\ -\frac{1}{2}\pi_A - \frac{1}{3}\pi_B + \pi_D = 0 \end{cases} \]
\[ \Rightarrow \pi_A = \frac{1}{4} \pi_B = \frac{3}{8} \pi_C = \frac{1}{8} \left[ \pi_D = \frac{1}{4} \right] \]
Prob 1.8 Solution.

(1) Compute $P^2$, we see $P_{00}^{(2)} = \frac{1}{7} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} = \frac{5}{21} > 0$

and $P_{12}^{(2)} > 0$ (1 → 3 → 2) 2 steps:

$P_{21}^{(1)} > 0$ (2 → 3 → 1)

$P_{13}^{(2)} > 0$ (1 → 2 → 3)

$P_{31}^{(1)} > 0$ (3 → 2 → 1)

$P_{14}^{(2)} > 0$ (1 → 3 → 4)

$P_{41}^{(2)} > 0$ (4 → 3 → 1)

$P_{35}^{(2)} > 0$ (3 → 4 → 5)

$P_{53}^{(2)} > 0$ (5 → 4 → 3)

Based on the above info, we know $P^2$ is regular, therefore $P$ is also regular

(Why? $P^2$ regular ⇒ there exists $n_0 \in N$ such that $(P^2)^{n_0} = P^{2n_0}$ all entries > 0 ⇒ $P$ regular)

(2) Let $\pi = (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5)$ where $\pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 = 1$

we have

$\pi (I - P) = 0 \Rightarrow\begin{cases} \pi_1 - \frac{1}{2} \pi_2 - \frac{1}{3} \pi_3 - \frac{1}{2} \pi_5 = 0 \\ -\frac{1}{2} \pi_1 + \pi_2 - \frac{1}{3} \pi_3 = 0 \\ -\frac{1}{2} \pi_1 - \frac{1}{2} \pi_2 + \pi_3 - \frac{1}{2} \pi_4 = 0 \\ -\frac{1}{2} \pi_3 + \pi_4 - \frac{1}{2} \pi_5 = 0 \\ -\frac{1}{2} \pi_4 + \pi_5 = 0 \end{cases}$

$\Rightarrow \pi_1 = \frac{22}{87}, \pi_2 = \frac{20}{87}, \pi_3 = \frac{27}{87}, \pi_4 = \frac{12}{87}, \pi_5 = \frac{6}{87}$
Prob. 1:13. Sol: Let $\pi = (\pi_0, \pi_1, \pi_2)$ where $\pi_0 + \pi_1 + \pi_2 = 1$.

We have
\[
\pi (I - P) = 0 \Rightarrow \begin{cases} 
0.6 \pi_0 - 0.6 \pi_1 - 0.4 \pi_2 = 0 \\
-0.4 \pi_0 + 0.8 \pi_1 - 0.2 \pi_2 = 0 \\
-0.2 \pi_0 - 0.2 \pi_1 + 0.6 \pi_2 = 0 
\end{cases}
\]

\[\Rightarrow \pi_0 = \frac{11}{24}, \quad \pi_1 = \frac{7}{24}, \quad \pi_2 = \frac{6}{24} = \frac{1}{4}\]

\[
\lim_{n \to \infty} P\{X_{n+1} = 2 | X_n = 1\} = \lim_{n \to \infty} \frac{P\{X_n = 1 | X_{n-1} = 2\} P\{X_{n-1} = 2\}}{P\{X_n = 1\}}
\]

\[
\ast \quad \frac{P_{21} \cdot \pi_2}{\pi_1} = \frac{0.2 \times \frac{1}{4}}{\frac{7}{24}} = \frac{6}{25}
\]

\[
\ast \quad \text{we know the long run distribution is independent of where the Markov chain starts.}
\]

That is
\[
P(X_{n-1} = 2) = P(X_{n-1} = 2 | X_0 = 0) = \pi_2
\]
\[
P(X_{n-1} = 1) = P(X_{n-1} = 1 | X_0 = 0) = \pi_1
\]
Ex 3.1. Sol: Let's draw a graph of this Markov chain:

\[
\begin{array}{c}
\overset{1}{\Box} \\
\overset{1}{\Box} \rightarrow \overset{1}{\Box} \rightarrow \overset{1}{\Box} \rightarrow \overset{1}{\Box} \rightarrow \overset{1}{\Box} \\
1 \quad 1 \\
\overset{1}{\Box} \leftrightarrow \overset{1}{\Box} \\
\end{array}
\]

From this graph we see, starting at 0, for every 5 steps \((0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 0)\), 8 steps \((0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 0)\) or their linear combination steps, this chain has positive probability of getting back to 0. Therefore, for \(n = 1, 2, \ldots, 20\),

\[
\begin{array}{c}
P_{00}^{(5)}, P_{00}^{(8)}, P_{00}^{(10)}, P_{00}^{(13)}, P_{00}^{(15)}, P_{00}^{(16)}, P_{00}^{(18)}, P_{00}^{(20)}\end{array}
\]

are positive, and the period of this M.C. is 1

(since \((8, 5) = 1) \#

Ex 3.4. Sol: Let's draw a graph of this M.C.:

From this graph we see there are 3 communicating classes:

<table>
<thead>
<tr>
<th>States</th>
<th>Period</th>
</tr>
</thead>
<tbody>
<tr>
<td>0, 3</td>
<td>1</td>
</tr>
<tr>
<td>2, 3, 4, 5, 3</td>
<td>1</td>
</tr>
<tr>
<td>1, 3</td>
<td>0</td>
</tr>
</tbody>
</table>