HW 2

Ex. 3.1. Sol: by inventory model, we have

\[ X_{n+1} = \begin{cases} X_n - 3 & \text{if } 0 < X_n \leq 3 \\ 3 - 3^{X_n} & \text{if } X_n \leq 0 \end{cases} \]

So \[ P_{ij} = P\{X_{n+1} = j \mid X_n = i\} \]

\[ = \begin{cases} P(3^{X_n} = i-j) & \text{if } 0 < i \leq 3 \\ P(3^{X_n} = 3-j) & \text{if } i \leq 0 \end{cases} \]

where \(-1 \leq i-j \leq 3\)

Therefore \[
P = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.3 & 0.3 & 0.4 \\
0.3 & 0.3 & 0.4 & 0 & 0 \\
0 & 0.3 & 0.3 & 0.4 & 0 \\
0 & 0 & 0.3 & 0.3 & 0.4 \\
\end{bmatrix}
\]

Ex. 3.2.

Sol.

\[ P_{i\hat{i}+1} = P(\hat{X}_{n+1} = \hat{i}+1 \mid \hat{X}_n = \hat{i}) = P \cdot \frac{N-\hat{i}}{N} \]

\[ \leq \text{Prob. of a ball chosen from } B \text{ and put into } A \]

\[ P_{i\hat{i}-1} = P(\hat{X}_{n+1} = \hat{i}-1 \mid \hat{X}_n = \hat{i}) = \frac{\hat{i}}{N} \]

\[ \leq \text{Prob. of a ball chosen from } A \text{ and put into } B \]

\[ P_{\hat{i}\hat{i}} = P(\hat{X}_{n+1} = \hat{i} \mid \hat{X}_n = \hat{i}) = \frac{\hat{i}}{N} + q \cdot \frac{N-\hat{i}}{N} = \hat{i} + (P-q) \cdot \frac{\hat{i}}{N} \]

Prob. of a ball chosen from A, put into A

Prob. of a ball chosen from B, put into B

Therefore, the transition matrix is

\[
P = \begin{bmatrix}
\frac{q}{N} & p & \frac{P-q}{N} & P - \frac{P}{N} & 0 \\
\frac{q}{N} & \frac{q}{N} & \frac{q}{N} & \frac{q}{N} & \frac{q}{N} \\
\frac{q}{N} & \frac{q}{N} & \frac{q}{N} & \frac{q}{N} & \frac{q}{N} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{q}{N} & \frac{q}{N} & \frac{q}{N} & \frac{q}{N} & \frac{q}{N} \\
\end{bmatrix}
\]
Prob. 3.3

Sol. (a) Since “the unfulfilled demand is not back ordered but is lost,” we have

\[ X_{n+1} = \begin{cases} 
(X_n - \frac{3}{n+1})_+ & \text{if } 0 < X_n \leq 2 \\
2 - \frac{3}{n+1} & \text{if } X_n = 0
\end{cases} \]

From this, we obtain the trans. matrix:

\[ P = \begin{pmatrix}
0 & 1 & 2 \\
0.1 & 0.4 & 0.5 \\
0.5 & 0.5 & 0 \\
0.1 & 0.4 & 0.5
\end{pmatrix} \]
Problem 34. Solution: First let’s recall geometric probability has the so-called “Memorylessness” property, that is:

\[ P(Z > t + s \mid Z > s) = P(Z > t) \]

which means that the conditional probability that we need to wait another \( t \) time before the first arrival, given that the first arrival has not yet happen after \( s \) time, is equal to the probability that we need to wait more than \( t \) time for the first arrival.

Back to our problem, it means that “knowing how long one customer has already been served does not provide any information to figure out if the customer can finish the service in \( t \) step or not.”

Let’s define a new random variable \( Y_n \).

\[ Y_n = \begin{cases} 1 & \text{if the customer being served at time } n \text{ will finish the service in one step} \\ 0 & \text{if the customer being served at time } n \text{ will finish the service in more than one step} \end{cases} \]

So we have \( P(Y_n = 0) = P(Z > 1) = 1 - P(Z = 1) = 1 - \alpha \)

and \( P(Y_n = 1) = P(Z = 1) = \alpha \)

therefore, we have

\[ X_{n+1} = (X_n - Y_n)^+ + Z_n \]

where \( Z_n \)’s are i.i.d. Bernoulli distribution, i.e.

\[ P(Z_1 = 0) = 1 - \beta \quad P(Z_1 = 1) = \beta \]
and $y_n$'s and $\bar{z}_n$'s are independent

From this, we can calculate the transition prob.:

For $i \geq 1$  
\[ P_{i,i+1} = P(X_{n+1} = i+1 \mid X_n = i) = P(Y_n = 0, \bar{z}_n = 1) \]
\[ = P(Y_n = 0) P(\bar{z}_n = 1) = (1-\alpha) \beta \]

indep. of $Y_n, \bar{z}_n$  
\[ P_{i,i} = P(X_{n+1} = i \mid X_n = i) = P(Y_n = 1, \bar{z}_n = 0) \]
\[ = P(Y_n = 1) P(\bar{z}_n = 0) = \alpha (1-\beta) \]

For $i = 0$, we have
\[ P_{0,1} = P(X_{n+1} = 1 \mid X_n = 0) = P(\bar{z}_n = 1) = \beta \]
\[ P_{0,0} = P(X_{n+1} = 0 \mid X_n = 0) = P(\bar{z}_n = 0) = 1-\beta \]

Other cases are all impossible to happen.

Therefore, the transition matrix is
\[
P = \begin{bmatrix}
1-\beta & \beta & 0 & 0 & 0 & \cdots \\
\alpha (1-\beta) & \alpha \beta + (1-\alpha)1 & \beta (1-\alpha) & 0 & 0 & \cdots \\
0 & \alpha (1-\beta) & \alpha \beta + (1-\alpha)1 & \beta (1-\alpha) & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
\end{bmatrix}
\]
Prob. 3.8

Sol. 

\[ P_{k,k+1} = P(X_{t+1} = k+1 \mid X_t = k) = q \cdot \frac{k}{N} \]

\[ P_{k,k-1} = P(X_{t+1} = k-1 \mid X_t = k) = p \cdot \frac{N-k}{N} \]

\[ P_{k,k} = P(X_{t+1} = k \mid X_t = k) = q \cdot \frac{N-k}{N} + p \cdot \frac{k}{N} \]