INTERSECTION LOCAL TIMES: LARGE DEVIATIONS AND LAWS OF THE ITERATED LOGARITHM

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Dedicated to Professor Lin, Zhengyan on his 65th birthday

Sample path intersection has been of interest to physicists for many years, due to its connections to renormalization group methods for quantum field theory, self-avoiding random walks and random polymers. In particular, the large deviations arising from sample path intersections have been applied to solve some hard problems raised by physicists such as the identification of certain critical exponents in polymer models and the investigation of intermittency in parabolic Anderson models. On the mathematical side, the study of the large deviations provides tail estimates needed for the strong laws in probability such as the law of the iterated logarithm. In this article we present a survey of some developments on the large deviations for intersection local times and related models.

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1. Introduction.

Let $W(t)$ be a d-dimensional Brownian motion. Assume for a moment that for any integer $p\geq 2$ we can define the quantity

$$\beta\big([0,t]^p_<\big)=\int\cdots\int_{\{0\leq s_1<\cdots<s_p\leq t\}}\prod_{j=2}^p\delta_0\big(W(s_j)-W(s_{j-1})\big)ds_1\cdots ds_p \tag{1.1}$$

where the notation $[0,t]^p_<$ represents the set

$$[0,t]^p_<=\big\{(s_1,\cdots,s_p)\in[0,t]^p;\quad s_1<\cdots<s_p\big\}.$$

Then $\beta\big([0,t]^p_<\big)$ measures the amount of p-multiple self-intersection of the random path $\{W(s);\ 0\leq s\leq t\}$. When $d=1$, we have

$$\beta\big([0,t]^p_<\big)=\frac{1}{p!}\int_{\mathbb{R}}L^p(t,x)dx \tag{1.2}$$

where $L(t,x)$ is the local time of the 1-dimensional Brownian motion $W(t)$. When $d\geq 2$, we will soon find out that $\beta\big([0,t]^p_<\big)$ can not be defined properly. For our convenience, however, we continue our discussion based on the false information that $\beta\big([0,t]^p_<\big)$ is properly defined whenever needed and we will take care of the difficulty later in section 4.

Let $W_1(t),\cdots,W_p(t)$ be independent d-dimensional Brownian motions. For any $t_1,\cdots,t_p>0$, the quantity formally written as

$$\alpha\big([0,t_1]\times\cdots\times[0,t_p]\big)=\int_0^{t_1}\cdots\int_0^{t_p}\prod_{j=2}^p\delta_0\big(W_{j-1}(s_{j-1})-W_j(s_j)\big)ds_p\cdots ds_1 \tag{1.3}$$

measures the amount of mutual intersection of the independent paths

$$\{W_1(s);\ 0\leq s\leq t_1\},\cdots,\{W_p(s);\ 0\leq s\leq t_p\}$$

In literature, the quantities $\beta\big([0,t]^p_<\big)$ and $\alpha\big([0,t_1]\times\cdots\times[0,t_p]\big)$ are called, respectively, p-multiple self-intersection local time and p-multiple mutual intersection local time.

One can also consider the intersection local times in the setting of random walks. Given a $\mathbb{Z}^d$-valued random walk $\{S(n)\}$, the p-multiple intersection local time is defined as

$$Q_n=\sum_{1\leq k_1<\cdots<k_p\leq n}1_{\{S(k_1)=\cdots=S(k_p)\}}. \tag{1.4}$$

Let $\{S_1(n)\},\cdots,\{S_p(n)\}$ be independent copies of $\{S(n)\}$. The mutual intersection local time run by $\{S_1(n)\},\cdots,\{S_p(n)\}$ is defined as

$$I_n = \sum_{k_1,\cdots,k_p=1}^{n} 1_{\{S_1(k_1)=\cdots=S_p(k_p)\}}. \tag{1.5}$$

Some closely related random quantities are the range

$$\#\big\{S(0,n]\big\} = \#\{S(1),\cdots,S(n)\} \tag{1.6}$$

and the intersection of the ranges

$$\begin{aligned} J_n &= \#\big\{S_1(0,n]\cap\cdots\cap S_p(0,n]\big\} \\ &= \big\{x\in\mathbb{Z}^d;\quad x=S_1(k_1)=\cdots=S_p(k_p) \text{ for some } k_1,\cdots,k_p\in(0,n]\big\}. \end{aligned} \tag{1.7}$$

The study of intersection local times is partially motivated by the needs from physics. Some problems in this area were suggested by physicists interested in applications to Euclidean quantum field theory (see Fernández, Fröhlich and Sokal (1992)), the growth of polymers (see ven der Hofstad and König (2001), den Hollander (1996), Vanderzande (1998), Westwater (1980, 1981, 1982)) and the polaron problem (see Donsker and Varadhan (1981, 1983), Mansmann (1991)). In the model of random walk in random environment, the dependence of a moving particle and a random environment frequently comes from the particle's ability to revisit sites with an attractive environment, and therefore measures of self-intersection quantify the degree of dependence between the movement and environment. We give the following partial list of literature on this topic: Carmona and Mochanov (1994), Cranston, Mountford and Shiga (2005), Ganternt, König and Shi (2007), van der Hofstad, König and Mörters (2006).

In this survey paper, we focus our attention on the large deviations for intersection local times and related models. More specifically, we are interested in the probabilities that the intersection local times take large values. Such probabilities are called tail probabilities in literature and often have exponential decay rates. Most of tail probabilities discussed in this article lead to the law of the iterated logarithm through some standard (or nearly standard) procedure of Borel-Cantelli lemma. Indeed, the relation between the tail probability and the law of the iterated logarithm is so close, that we feel there is no need to distinguish one from another in our following discussion.

In addition to its application to the law of the iterated logarithm, the large deviations for intersection local times connect to the problems in some other disciplines. We list three connections below.

Connection to the study of polymer: Physicists concern about geometric shape of the polymer which is often described by a suitable random path (such as a Brownian curve). The geometry of a polymer is decided by the intensity that the random path intersects itself. The case when $\beta\big([0,t]<^2\big)$ is large corresponds to a "contracting" polymer; while the case when $\beta\big([0,t]<^2\big)$ is small corresponds to a "spread-out" polymer. The geometric shape of a polymer is often influenced by the environment (media). If the environment encourages attraction among the molecules, then the polymer is contracting. In this case, the polymer is called self-attracting polymer, In the opposite case, the polymer is spread-out and is called self-repelling polymer.

In physics, the probability measure $\widehat{P}_\lambda$ on $C\big\{[0,t],\mathbb{R}^2\big\}$ defined as

$$\widehat{P}_\lambda(A)=\widehat{C}_\lambda^{-1}\mathbb{E}\exp\bigg\{\lambda\int\!\!\int_{\{0\le r<s\le t\}}\delta_0\big(W(r)-W(s)\big)drds\bigg\}1_{\{W(\cdot)\in A\}} \tag{1.9}$$

is regarded as the distribution of the curve representing a self-attracting polymer, where $A\subset C\big\{[0,t],\mathbb{R}^2\big\}$, $\lambda>0$ represents the temperature and $\widehat{C}_\lambda$ is the normalizing constant making $\widehat{P}_\lambda$ a probability measure. Indeed, the curves with high self-intersection are given more distributional weight and the degree of this favoritism is decided by the value of λ.

Similarly, a self-repelling polymer is modeled by the distribution

$$\widetilde{P}_\lambda(A)=\widetilde{C}_\lambda^{-1}\mathbb{E}\exp\bigg\{-\lambda\int\!\!\int_{\{0\le r<s\le t\}}\delta_0\big(W(r)-W(s)\big)drds\bigg\}1_{\{W(\cdot)\in A\}}. \tag{1.10}$$

One of the problems that physicists try to understand is the behaviors of the partition function

$$\mathbb{E}\exp\bigg\{\pm\lambda\int\!\!\int_{\{0\le r<s\le t\}}\delta_0\big(W(r)-W(s)\big)drds\bigg\}.$$

As we will see in section 4, the large deviations is the right tool to the investigations of this problem.

Connection to Parabolic Anderson model: Parabolic Anderson model is described by the system

$$\begin{cases}\partial_t u(t,x)=\kappa\triangle u(t,x)+\xi(x)u(t,x) & (t,x)\in(0,\infty)\times\mathbb{Z}^d\ (\text{or }(0,\infty)\times\mathbb{R}^d)\\ u(0,x)=1 & x\in\mathbb{Z}^d\ (\text{or }\mathbb{R}^d)\end{cases}$$

where $\xi(x)$ is a random field. For example, one can take $\xi(x)$ ($x\in\mathbb{Z}^d$) as i.i.d. random variables; or one can take $\xi(x)=\dot{W}(x)$ with $W(x)$ ($x\in\mathbb{R}^d$) being a Brownian sheet. We refer the reader to the references Carmona and Mochanov (1994), Cranston, Mountford and Shiga (2005), van der Hofstad, König and Mörters (2006).

In a parabolic Anderson model, the quantity $u(t,x)$ is interpreted as expected total mass at time t carried by a particle initially placed at the site x with a unit mass on it. The particle diffuses like a simple random walk (discrete setting) or like a Brownian motion (continuous setting) with generator $\kappa\triangle$; when present at site x, its mass is increased/decreased by an infinitesimal amount at rate $\pm\xi(x)\vee 0$. Of particular interest is the phenomenon of *intermittency*, which is often reflected by the asymptotic order of the quantity

$$\log\langle u^p(t,0)\rangle$$

as $t\to\infty$, where the notation "$\langle\cdot\rangle$" is used for the expectation with respect to the random potential $\xi(x)$. In the case $\xi(x)=\dot{W}(x)$ with $W(x)$ $(x\in\mathbb{R})$ being a 1-dimensional Brownian motion, for example, by Feynman-Kac formula we have

$$\langle u^p(t,0)\rangle=\mathbb{E}\exp\left\{\frac{\kappa}{2}\int_0^t\dot{W}\big(B(s)\big)ds\right\}=\mathbb{E}\exp\left\{\frac{\kappa^2}{8}\int_{\mathbb{R}}L^2(t,x)dx\right\}.$$

So the problem is reduced to the tail behaviors of intersection local times.

Random walk in random scenery: The asymptotic behavior of self-intersection local times contributes in a fundamental way to the limit theorems for the random walks in random sceneries. We refer the reader to Kesten and Spitzer (1979), Khoshnevisan and Lewis (1998), Csáki, König and Shi (1999), Révész and Shi (2000), Ganternt, König and Shi (2007), Asselah and Castell (2006) for the links to this problem. To established this connection, let $S(n)$ be a symmetric random walk taking values in $\mathbb{Z}^d$ and consider the random sequence

$$R_n=\sum_{k=1}^n\xi\big(S(k)\big)\qquad n=1,\cdots$$

where $\xi(x)$ $(x\in\mathbb{Z}^d)$ is often assumed to be an independent (of the random walks) family of i.i.d random variable with mean zero and strong enough integrability. In the case $p=1$, this model is known as random walk in random scenery. Let $l(n,x)$ be the local time of the random walk:

$$l(n,x)=\sum_{k=1}^n 1_{\{S(k)=x\}}\qquad x\in\mathbb{Z}^d\quad n=1,\cdots.$$

Notice that

$$R_n=\sum_{x\in\mathbb{Z}^d}\xi(x)l(n,x)\qquad n=1,2,\cdots.$$

By the fact that conditionally on $l(n,x)$, R_n has the variance given by the quadratic form

$$\sum_{x\in\mathbb{Z}^d}l^2(n,x)$$

which is essentially twice of self-intersection local time of $\{S(n)\}$.

The systematic study of sample path intersection goes back at least to the work by Dvoretzky, Erdös and Kakutani (1950, 1954) on the multiple points of Brownian paths. The list of the references is too long to be included here. We mention some pioneering works by Varadhan (1969), Geman-Horowitz-Rosen (1984), Dynkin (1988), Yor (1985), Calais-Yor (1987), Le Gall (1986a, b), Rosen-Le Gall (1991) on the renormalization, construction of intersection local times, and (or) on the laws of weak convergence for the intersection local times and related models. The remarkable papers by Donsker-Varadhan (1975a, 1975b, 1976, 1979) on the large deviations created profound influence in the study of large deviations for intersection local times and related models. A key ingredient in their approach is Feynman-Kac Formula. A partial list of the references on this line are Mansmann (1991), Bolthausen (1999), van den Berg-Bolthausen-den Hollander (2001, 2004), Kesten-Hamana (2001, 2002), Bass-Chen (2004) Bass-Chen-Rosen (2006a, b), Chen (2004), Chen-Li (2004) and Chen-Li-Rosen (2005). In their paper on the upper tails for intersection local times, König and Mörters (2002) proposed a method computing high moment of intersection local times. This approach started to show its effectiveness in dealing with some difficult problems in the area of large deviations for intersection local times and for some related models. We refer the reader to section 7 for discussion on this method and for some applications.

We mention some remarkable contributions from Xianyin Zhou (1963-1996), whose life was cut short by sudden disease. In his short academic career, Zhou produced more than 50 high quality papers. Significant part of his work deals with sample path intersection and its connection to polymers. We refer the reader to a special memorial volume (Zhou (2002)) for a collection of Zhou's work.

The literature is vast and one has to make a selection of topics. What is presented here does reflect the author's interest and preference. Apart from the topics addressed in this article, we mention the works in the some recent important development in the closely related areas. (a).The solution of the intersection exponent problem and its associations with self-avoiding random walks and with the stochastic Loewner evolution: The reader is referred to the papers by Lawler, Scheraman and Werner (2001a, 2001b, 2002), the book by Lawler (2005). (b). The large deviations for the random walks and Brownian motions under the polymer structure in the spirit of (1.9) and (1.10): The reader is referred to van der Hofstad, den Hollander and König (2003). (c). The large deviations for Brownian intersection local times confined in a bounded domain and running up to the exit times of the domain and, their applications to the problem of finding the Hausdorff dimension spectrum for the thick points of the intersection: The reader is referred to König and Mörters (2002).

Little has been known beyond the models with independent increments. To the best of the author's knowledge, the large deviations for the intersection local times of general Markov processes remain essentially open. In this category, the models of particular interest are diffusion processes, super processes and random walks and Brownian motions on graphs and fractals. Perhaps an even harder problem is to ask the same question to the non-Markovian processes such as fractional Brownian motions and Gaussian processes in general. Here we refer the reader to the paper by Hu and Nualart (2005) for renormalization of the self-intersection local times of fractional Brownian motions, and to the paper by Xiao (2006) for the recent development of the method known as *local non-determinism* and its applications on the tail probabilities for local times and intersection local times of Gaussian processes.

We end this introduction with an outline of the rest of the paper. In section 2 we investigate exponential asymptotics for the smoothed intersection local times by Feynman-Kac formula. In section 3, we discuss the large deviations for the mutual intersection local time $\alpha([0,t]^p)$ run by independent Brownian motions and for the local times of additive Brownian motions. In section 4, we summarize the large deviations for self-intersection local time of a single Brownian. In section 5 and 6, our attention is switched to the intersections of random walks in lattice spaces. In section 7, we discuss a new method in computing high moment of intersection local times.

2. Exponential asymptotics for smoothed intersection local times.

A substantial amount of our work is fighting against singularity, which increases with the dimension. When $d=1$, the intersection local times can be written in terms of the local times, as we have seen in (1.2). When $d\ge 2$, the local time no longer exist. We introduce

$$L(t,x,\epsilon)=\int_0^t h_\epsilon\big(W(s)-x\big)ds$$

instead, where

$$h_\epsilon(x)=\epsilon^{-d}h(\epsilon^{-1}x)\qquad x\in\mathbb{R}^d,\quad \epsilon>0$$

and $h(x)$ is a suitable probability density on $\mathbb{R}^d$. Our strategy is to establish exponential asymptotics for the smoothed quantities

$$\int_{\mathbb{R}^d}L^p(t,x,\epsilon)dx,\qquad \int_{\mathbb{R}^d}\prod_{j=1}^p L_j(t,x,\epsilon)dx$$

and to approximate intersection local times using these quantities by letting $\epsilon\to 0^+$. To this end we write, for $\theta>0$,

$$M_\epsilon(\theta)=\sup_{f\in\mathcal{F}_d}\left\{\theta\bigg(\int_{\mathbb{R}^d}\big[(f^2*h_\epsilon)(x)\big]^p dx\bigg)^{1/p}-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla f(x)|^2dx\right\}\tag{2.1}$$

$$N_\epsilon(\theta) = \sup_{f\in\mathcal{F}_d} \left\{ \theta\left(\int_{\mathbb{R}^d} \big[(f^2 * h_\epsilon)(x)\big]^p dx \right)^{1/p} - \frac{p}{2} \int_{\mathbb{R}^d} |\nabla f(x)|^2 dx \right\} \tag{2.2}$$

where

$$\mathcal{F}_d = \big\{ f \in \mathcal{L}^2(\mathbb{R}^d); \quad ||f||_2 = 1 \text{ and } \nabla f \in \mathcal{L}^2(\mathbb{R}^d) \big\}. \tag{2.3}$$

The following result is given in Chen (2004).

Theorem 2.1. *For any $\theta > 0$ and $p \geq 2$,*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} \exp\left\{ \theta\left(\int_{\mathbb{R}^d} L^p(t, x, \epsilon) dx \right)^{1/p} \right\} = M_\epsilon(\theta) \tag{2.4}$$

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} \exp\left\{ \theta\left(\int_{\mathbb{R}^d} \prod_{j=1}^p L_j(t, x, \epsilon) dx \right)^{1/p} \right\} = N_\epsilon(\theta). \tag{2.5}$$

One of the main ingredients in our approach for Theorem 2.1 is the well known Feynman-Kac formula, which claim that for a nice function $f \in \mathbb{R}^d$, the semi-group $\{T_t\}_{t\geq 0}$ of self-adjoint linear operators on $\mathcal{L}^2(\mathbb{R}^d)$

$$T_t g(x) = \mathbb{E}_x \exp\left\{ \int_0^t f\big(W(s)\big) ds \right\} g\big(W(t)\big), \qquad g \in \mathcal{L}^2(\mathbb{R}^d)$$

has infinitesimal generator

$$\mathcal{A} g(x) = \frac{1}{2} \triangle g(x) + f(x) g(x).$$

The relation is formally given as $T_t = e^{t\mathcal{A}}$. Consequently,

$$\begin{aligned} &\mathbb{E} \exp\left\{ \int_0^t f\big(W(s)\big) ds \right\} \approx \sup_{g\in\mathcal{F}_d} \langle g, T_t g \rangle = \exp\big\{ t \sup_{g\in\mathcal{F}_d} \langle g, \mathcal{A} g \rangle \big\} \\ &= \exp\left\{ t \sup_{g\in\mathcal{F}_d} \left(\int_{\mathbb{R}^d} f(x) g^2(x) dx - \frac{1}{2} \int_{\mathbb{R}^d} |\nabla g(x)|^2 dx \right) \right\}. \end{aligned}$$

Thus

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} \exp\left\{ \int_0^t f\big(W(s)\big) ds \right\} \\ &= \sup_{g\in\mathcal{F}_d} \left\{ \int_{\mathbb{R}^d} f(x) g^2(x) dx - \frac{1}{2} \int_{\mathbb{R}^d} |\nabla g(x)|^2 dx \right\}. \end{aligned} \tag{2.6}$$

The reader is referred to Remillard (2000) for the details of the proof of (2.6).

Let $q > 1$ be the conjugate number of p defined by $p^{-1} + q^{-1} = 1$. By Hölder inequality, for any f with $|f||_q = 1$

$$\left(\int_{\mathbb{R}^d} L^p(t, x, \epsilon) dx \right)^{1/p} \geq \int_{\mathbb{R}^d} L(t, x, \epsilon) f(x) dx = \int_0^t (f * h_\epsilon)\big(W(s)\big) ds.$$

Replacing f by $\theta f * h_\epsilon$ in (2.6),

$$\begin{aligned}
&\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\theta\Big(\int_{\mathbb{R}^d}L^p(t,x,\epsilon)dx\Big)^{1/p}\right\}\\
&\ge \sup_{g\in\mathcal{F}_d}\left\{\theta\int_{\mathbb{R}^d}(f*h_\epsilon)(x)g^2(x)dx-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}\\
&= \sup_{g\in\mathcal{F}_d}\left\{\theta\int_{\mathbb{R}^d}f(x)(g^2*h_\epsilon)(x)dx-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla g(x)|^2dx\right\}.
\end{aligned}$$

Taking supremum over $||f||_q=1$ on the right hand side gives the lower bound for (2.4).

The upper bound for (2.4) follows, in spirit, also from the Feynman-Kac large deviation given in (2.6). The argument is much harder than the one for the lower bound and the technical difficulty here is the absence of exponential tightness of the model.

The upper bound of (2.5) follows from the deterministic relation

$$\Big(\int_{\mathbb{R}^d}\prod_{j=1}^p L_j(t,x,\epsilon)dx\Big)^{1/p}\le\frac{1}{p}\sum_{j=1}^p\Big(\int_{\mathbb{R}^d}L_j^p(t,x,\epsilon)dx\Big)^{1/p} \tag{2.7}$$

and (2.4). The idea behind the lower bound of (2.5) is that the inequality (2.7) is essentially reversible on the event

$$\Big\{L_1(t,\cdot,\epsilon)\approx\cdots\approx L_p(t,\cdot,\epsilon)\Big\}$$

and that decay of the probability of this set is slower than exponential rate.

3. Intersection of independent Brownian motions.

Let $W_1(t),\cdots,W_p(t)$ be independent d-dimensional Brownian motions. If we allow $W_j(\cdot)$ run up to time t_j $(j=1,\cdots,p)$, a natural question is to ask how much time is spent for the p independent trajectories $W_1(t),\cdots,W_p(t)$ to intersect. In other words, we are interested in the time set

$$\Big\{(s_1,\cdots,s_p)\in[0,t_1]\times\cdots\times[0,t_p];\quad W_1(s_1)\approx\cdots\approx W_p(s_p)\Big\}.$$

If properly defined, the Lebesgue measure of this set is called the intersection local time of $W_1(t),\cdots,W_p(t)$ and is denoted by $\alpha\big([0,t_1]\times\cdots\times[0,t_p]\big)$.

One of the fascinating phenomena in the area of sample path intersection is dimension dependence. It is well known (Dvoretzky-Erdös-Kakutani (1950, 1954)) that

$$W_1(0,\infty)\cap\cdots\cap W_p(0,\infty)\ne\phi$$

if and only if $p(d-2) < d$. That is, it is hard for trajectories to intersect in high dimensional space.

Under $p(d-2) < d$, there are two equivalent ways to construct Brownian intersection local time in the multi-dimensional case. The first approach (Geman, Horowitz and Rosen (1984)) corresponds to the notation given in (1.3). Geman, Horowitz and Rosen (1984) proved that under $p(d-2) < d$, the occupation measure on $\mathbb{R}^{d(p-1)}$ given by

$$\mu_A(B) = \int_A 1_B\big(W_1(s_1) - W_2(s_2), \cdots, W_{p-1}(s_{p-1}) - W_p(s_p)\big) ds_1 \cdots ds_p \qquad B \subset \mathbb{R}^{d(p-1)}$$

is absolutely continuous, with probability 1, with respect to Lebesgue measure on $\mathbb{R}^{d(p-1)}$ for any Borel set $A \subset (\mathbb{R}^p)^+$ (in particular, for $A = [0,t_1] \times \cdots \times [0,t_p]$) and, the density $\alpha(x, A)$ of such measure can be chosen in such a way that the function

$$(x, t_1, \cdots, t_p) \longmapsto \alpha\big(x, [0,t_1] \times \cdots \times [0,t_p]\big) \qquad x \in \mathbb{R}^{d(p-1)} \quad (t_1, \cdots, t_p) \in (\mathbb{R}^p)^+$$

is jointly continuous. The random measure $\alpha(\cdot)$ on $(\mathbb{R}^p)^+$ is defined as

$$\alpha(A) = \alpha(0, A) \qquad \forall \text{ Borel set } A \subset (\mathbb{R}^p)^+.$$

Another approach (Le Gall (1992)) constitutes the notation

$$\alpha\big([0,t_1] \times \cdots [0,t_p]\big) = \int_{\mathbb{R}^d} \left[\prod_{j=1}^p \int_0^{t_j} \delta_0\big(W(s) - x\big) ds\right] dx. \tag{3.1}$$

Let $f(x)$ be a nice probability density function on $\mathbb{R}^d$. Given $\epsilon > 0$, write $f_\epsilon(x) = \epsilon^{-d} f(\epsilon^{-1}x)$ and define

$$\alpha_\epsilon\big([0,t_1] \times \cdots \times [0,t_p]\big) = \int_{\mathbb{R}^d} \left[\prod_{j=1}^p \int_0^{t_j} f_\epsilon\big(W(s) - x\big) ds\right] dx.$$

Under $p(d-2) < d$, Le Gall (1992) proved that there is a random variable $\alpha\big([0,t_1] \times \cdots \times [0,t_p]\big)$ such that

$$\lim_{\epsilon \to 0^+} \alpha_\epsilon\big([0,t_1] \times \cdots \times [0,t_p]\big) = \alpha\big([0,t_1] \times \cdots \times [0,t_p]\big)$$

holds in L^m-norm for any $m \ge 1$ and for any $t_1, \cdots, t_p > 0$.

By the scaling property of Brownian motions

$$\alpha\big([0,t]^p\big) \stackrel{d}{=} t^{\frac{2p-d(p-1)}{2}} \alpha\big([0,1]^p\big). \tag{3.2}$$

In the special case $d = 1$, let $L_1(t,x), \cdots, L_p(t,x)$ be the local times of $W_1, \cdots, W_p$, respectively. By Le Gall's construction, one can see that

$$\alpha\big([0,t_1] \times \cdots \times [0,t_p]\big) = \int_{-\infty}^{\infty} \prod_{j=1}^p L_j(t,x) dx.$$

The following theorem is given in Chen (2004).

Theorem 3.1. *Under $p(d-2) < d$,*

$$\lim_{t\to\infty} t^{-\frac{2}{d(p-1)}} \log \mathbb{P}\Big\{\alpha\big([0,1]^p\big) \geq t\Big\} = -\frac{p}{2}\kappa(d,p)^{-\frac{4p}{d(p-1)}} \tag{3.3}$$

where $\kappa(d,p)$ is the best constant of the Gagliardo-Nirenberg inequality

$$||f||_{2p} \leq C||\nabla f||_2^{\frac{d(p-1)}{2p}} ||f||_2^{1-\frac{d(p-1)}{2p}} \qquad f \in W^{1,2}(\mathbb{R}^d) \tag{3.4}$$

and

$$W^{1,2}(\mathbb{R}^d) = \big\{f \in \mathcal{L}^2(\mathbb{R}^d); \quad \nabla f \in \mathcal{L}^2(\mathbb{R}^d)\big\}.$$

Finding the best constant of Gagliardo-Nirenberg inequalities remains open in general. It has been attracting considerable attention partially due to its connection to some problems in physics. The best constant for Nash inequality, which is a special case of Gagliardo-Nirenberg inequalities, was found by Carlen and Loss (1993). See also papers by Cordero-Erausquim, Nararet and Villani (2004) and by Del Pino and Dolbeault (2002) for recent progress on the best constants for a class of Gagliardo-Nirenberg inequalities. See the paper Del Pino and Dolbeault (2003) for a connection between the best constants for a class of Gagliardo-Nirenberg inequalities and logarithmic Sobolev inequalities. Two papers are directly related to $\kappa(2,3)$: Levine (1980) obtained the sharp estimate

$$\sqrt[3]{\frac{1}{4.6016}} < \kappa(2,3) < \sqrt[3]{\frac{1}{4.5981}}. \tag{3.5}$$

He conjectured that $\kappa(2,3) = \pi^{-4/9}$. Another numerical solution was obtained by Weinstein (1983). He obtained

$$\kappa(2,2) \approx \sqrt[4]{\frac{1}{\pi \times 1.86225\cdots}}. \tag{3.6}$$

Here is the basic idea behind Theorem 3.1: As $\epsilon \to 0^+$, $\alpha([0,t]^p)$ is so closed to

$$\int_{\mathbb{R}^d} \prod_{j=1}^p L_j(t,x,\epsilon)dx$$

given in Theorem 2.1 that (2.5) implies replaced by

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} \exp\Big\{\theta\Big(\alpha([0,t]^p)\Big)^{1/p}\Big\} \\
&= \sup_{f\in\mathcal{F}_d} \Big\{\theta\Big(\int_{\mathbb{R}^d} |f(x)|^{2p} dx\Big)^{1/p} - \frac{p}{2}\int_{\mathbb{R}^d} |\nabla f(x)|^2 dx\Big\} \\
&= \theta^{\frac{2p}{2p-d(p-1)}} p^{-\frac{d(p-1)}{2p-d(p-1)}} \sup_{f\in\mathcal{F}_d} \Big\{\Big(\int_{\mathbb{R}^d} |f(x)|^{2p} dx\Big)^{1/p} - \frac{1}{2}\int_{\mathbb{R}^d} |\nabla f(x)|^2 dx\Big\}
\end{aligned}$$

where the last step follows from a suitable rescaling.

Finally, Theorem 3.1 follows from scaling identity (2.10), a standard application of Gärtner-Ellis theorem on large deviation, and the relation (Lemma A.2, Chen (2004))

$$\sup_{f\in\mathcal{F}_d}\left\{\Big(\int_{\mathbb{R}^d}|f(x)|^{2p}dx\Big)^{1/p}-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla f(x)|^2dx\right\}$$
$$=\frac{2p-d(p-1)}{2p}\Big(\frac{d(p-1)}{p}\Big)^{\frac{d(p-1)}{2p-d(p-1)}}\kappa(d,p)^{\frac{4p}{2p-d(p-1)}}.$$

□

From the view point of Geman, Horowitz and Rosen (1984), the intersection local time $\alpha([0,t]^p)$ is the local time of the multi-parameter process

$$\widetilde{W}(t_1,\cdots,t_p)=\Big(W_1(t_1)-W_2(t_2),\cdots,W_{p-1}(t_{p-1})-W_p(t_p)\Big)$$

as indicated by (1.3). In this regard, Theorem 3.1 contributes solution to a problem on the tail probability for the local times of multi-parameter processes.

Perhaps the most important multi-parameter process is Brownian sheet. We refer the reader to the book by Khoshnevisan (2002) for the overall information about the Brownian sheets and multi-parameter processes, among which it has been known that a Brownian sheet $B(t_1,\cdots,t_p)$ with space dimension d and time dimension $p\geq 2$ has local time if and only if $d<2p$. The exact tail of the local time of Brownian sheet remains unknown — an indication how little we know about the tails of the local times of multi-parameter processes.

In the rest of the section, we discuss some recent progress on a model which resembles Brownian sheet in many ways but is more amenable to analysis than Brownian sheet. We hope it will give us more insight and sharpen our tools for the case of Brownian sheets.

The model is known as *additive Brownian motion* and is defined by

$$\widetilde{B}(t_1,\cdots,t_p)=W_1(t_1)+\cdots+W_p(t_p)\qquad t_1,\cdots,t_p\geq 0.$$

We concern about the large deviation problem for the local time

$$\eta^x(I)=\int_I\delta_x\big(W_1(s_1)+\cdots+W_p(s_p)\big)ds_1\cdots ds_p\qquad x\in\mathbb{R}^d,\quad I\subset(\mathbb{R}^+)^p$$

of $\widetilde{B}(t_1,\cdots,t_p)$ and rely on two recent papers by Khoshnevisan, Xiao and Zhong (2003a, b) for the constructions of the local time $\eta^x(I)$. In their papers, Khoshnevisan, Xiao and

Zhong (2003a, b) consider a more general multi-parameter random field named additive Lévy process, which is generated by independent Lévy processes. In their construction, $\eta^x(I)$ is defined as the density function of the occupation measure μ_I:

$$\mu_I(A) = \int_I \delta_{W_1(s_1)+\cdots+W_p(s_p)}(A)ds_1\cdots ds_p \qquad A\subset \mathbb{R}^d$$

in the case when μ_I is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$. Not surprising at all, the necessary and sufficient condition for existence of $\eta^x(I)$ is $d<2p$, same as the one for existence of the local time of Brownian sheet. Under this condition, the local time

$$\eta^x\big([0,t]^p\big) \qquad (x,t)\in\mathbb{R}^d\times\mathbb{R}^+$$

is jointly continuous in (x,t) (Corollary 3.3, Khoshnevisan, Xiao and Zhong (2003b)).

By Brownian scaling property,

$$\eta\big([0,t]^p\big) \stackrel{d}{=} t^{\frac{2p-d}{2}}\eta\big([0,1]^p\big). \tag{3.7}$$

The local time $\eta^x\big([0,t]^p\big)$ connects to the intersection local time $\alpha\big([0,1]^p\big)$ when $p=2$, in which case

$$\eta^0\big([0,t]^2\big) \stackrel{d}{=} \alpha\big([0,t]^2\big).$$

To state the large deviations for $\eta^x\big([0,t]^p\big)$, we define

$$\rho_1 = \sup_{||f||_2=1}\int_{\mathbb{R}^d}\bigg[\int_{\mathbb{R}^d}\frac{f(\lambda+\gamma)f(\gamma)}{\sqrt{1+2^{-1}|\lambda+\gamma|^2}\sqrt{1+2^{-1}|\gamma|^2}}d\gamma\bigg]^p d\lambda$$

$$\rho_2 = \sup_{||f||_2=1}\int_{\mathbb{R}^d}\bigg[\int_{\mathbb{R}^d}\frac{f(\lambda+\gamma)f(\gamma)}{\sqrt{1+2^{-1}|\lambda+\gamma|^2}\sqrt{1+2^{-1}|\gamma|^2}}d\gamma\bigg]^{2p} d\lambda.$$

By essentially Hölder inequality, one can prove that under $d<2p$

$$0<\rho_1\le\int_{\mathbb{R}^d}\frac{1}{\big(1+2^{-1}|\gamma|^2\big)^p}d\lambda<\infty,$$

$$0<\rho_2\le\int_{\mathbb{R}^d}\frac{1}{\big(1+2^{-1}|\gamma|^2\big)^{2p}}d\lambda<\infty.$$

The following result is given in Chen (2006a, 2007a).

Theorem 3.2. *Under $d < 2p$,*

$$\lim_{t\to\infty} t^{-2/d}\log \mathbb{P}\Big\{\eta^0\big([0,1]^p\big)\ge t\Big\} = -(2\pi)^2\frac{d}{2}\Big(1-\frac{d}{2p}\Big)^{\frac{2p-d}{d}}\rho_1^{-2/d} \tag{3.8}$$

$$\lim_{t\to\infty} t^{-2/d}\log \mathbb{P}\Big\{\sup_{x\in\mathbb{R}^d}\eta^x\big([0,1]^p\big)\ge t\Big\} = -(2\pi)^2\frac{d}{2}\Big(1-\frac{d}{2p}\Big)^{\frac{2p-d}{d}}\rho_1^{-2/d} \tag{3.9}$$

$$\lim_{t\to\infty} t^{-2/d}\log \mathbb{P}\Big\{\int_{\mathbb{R}^d}\Big[\eta^x\big([0,1]^p\big)\Big]^2dx\ge t\Big\} = -\frac{d}{4}(2\pi)^2\Big(1-\frac{d}{4p}\Big)^{\frac{4p-d}{d}}\rho_2^{-2/d}. \tag{3.10}$$

Theorem 3.2 can be extended to the setting of additive Lévy processes and additive random walks. We refer the reader to Chen (2007b) for details.

It is interesting to see that $\sup\limits_{x\in\mathbb{R}^d}\eta^x([0,1]^p)$ has exactly same up tail as $\eta^0\big([0,1]^p\big)$. The proof of (3.10) is based on (3.8) and on a chaining argument. To explain the idea behind (3.8), we recall the following lemma given in König and Mörters (2002).

Lemma 3.1. *Let Y be any non-negative random variable and let $\theta > 0$ be fixed. Assume that*

$$\lim_{m\to\infty}\frac{1}{m}\log\frac{1}{(m!)^\theta}\mathbb{E}\,Y^m = -\kappa \tag{3.11}$$

for some $\kappa\in\mathbb{R}$. Then we have

$$\lim_{t\to\infty} t^{-1/\theta}\log\mathbb{P}\{Y\ge t\} = -\theta e^{\kappa/\theta}. \tag{3.12}$$

To prove (3.8), therefore, one needs to establish

$$\lim_{m\to\infty}\frac{1}{m}\log(m!)^{-d/2}\mathbb{E}\Big[\eta([0,1]^p)^m\Big] = \log\Big(\frac{2p}{2p-d}\Big)^{\frac{2p-d}{2}} + \log\frac{\rho_1}{(2\pi)^d}. \tag{3.13}$$

By Fourier transform, for any $t_1,\cdots,t_p\ge 0$,

$$\eta([0,t_1]\times\cdots\times[0,t_p]) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}d\lambda\prod_{j=1}^p\int_0^{t_j}\exp\Big\{i\lambda\cdot W_j(s)\Big\}ds.$$

For any integer $m\ge 1$, by time rearrangement and by increment independence,

$$\begin{aligned}&\mathbb{E}\Big[\eta^0([0,t_1]\times\cdots\times[0,t_p])^m\Big]\\ &=\frac{1}{(2\pi)^{dm}}\int_{(\mathbb{R}^d)^m}d\lambda_1\cdots d\lambda_m\prod_{j=1}^p\sum_{\sigma\in\Sigma_m}\\ &\times\int_{\{0\le s_1\le\cdots\le s_m\le t_j\}}\prod_{k=1}^m\exp\Big\{-\frac{s_k-s_{k-1}}{2}\Big|\sum_{j=1}^k\lambda_{\sigma(j)}\Big|^2\Big\}ds_1\cdots ds_m.\end{aligned} \tag{3.14}$$

where Σ_m is the permutation group on $\{1,\cdots,m\}$ and $s_0=0$.

To simplify the above representation, we replace $t_1\cdots,t_p$ by the i.i.d exponential times $\tau_1,\cdots,\tau_p$. We assume that the exponential distribution has parameter 1 and that $\tau_1,\cdots,\tau_p$ are independent of $W_1(t),\cdots,W_p(t)$. From (3.14)

$$\begin{aligned}
&\mathbb{E}\Big[\eta([0,\tau_1]\times\cdots\times[0,\tau_p])^m\Big]\\
&=\frac{1}{(2\pi)^{dm}}\int_{(\mathbb{R}^d)^m}d\lambda_1\cdots d\lambda_m\bigg[\sum_{\sigma\in\Sigma_m}\int_0^\infty e^{-t}dt\\
&\times\int_{\{0\le s_1\le\cdots s_m\le t\}}\prod_{k=1}^m\exp\Big\{-\frac{s_k-s_{k-1}}{2}\Big\vert\sum_{j=1}^k\lambda_{\sigma(j)}\Big\vert^2\Big\}ds_1\cdots ds_m\bigg]^p\\
&=\frac{1}{(2\pi)^{dm}}\int_{(\mathbb{R}^d)^m}d\lambda_1\cdots d\lambda_m\bigg[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m\int_0^\infty e^{-t}\exp\Big\{-\frac{t}{2}\Big\vert\sum_{j=1}^k\lambda_{\sigma(j)}\Big\vert^2\Big\}dt\bigg]^p\\
&=\frac{1}{(2\pi)^{dm}}\int_{(\mathbb{R}^d)^m}d\lambda_1\cdots d\lambda_m\bigg[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m Q\Big(\sum_{j=1}^k\lambda_{\sigma(j)}\Big)\bigg]^p
\end{aligned}\tag{3.15}$$

where $Q(\lambda)=(1+2^{-1}|\lambda|^2)^{-1}$.

A crucial step is to establish

$$\lim_{m\to\infty}\frac{1}{m}\log\frac{1}{(m!)^p}\int_{(\mathbb{R}^d)^m}d\lambda_1\cdots d\lambda_m\bigg[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m Q\Big(\sum_{j=1}^k\lambda_{\sigma(j)}\Big)\bigg]^p=\log\rho_1\tag{3.16}$$

which is left to the discussion in section 7.

Let $t_1,\cdots,t_p\ge 0$. By (3.14), by Hölder inequality and by the scaling (3.7),

$$\begin{aligned}
&\mathbb{E}\Big[\eta([0,t_1]\times\cdots\times[0,t_p])^m\Big]\\
&\le\prod_{j=1}^p\Big\{\mathbb{E}\Big[\eta\big([0,t_j]^p\big)^m\Big]\Big\}^{1/p}=(t_1\cdots t_p)^{\frac{2p-d}{2p}m}\mathbb{E}\Big[\eta^0\big([0,1]^p\big)^m\Big].
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathbb{E}\Big[\eta([0,\tau_1]\times\cdots\times[0,\tau_p])^m\Big]\\
&=\int_0^\infty\cdots\int_0^\infty e^{-(t_1+\cdots+t_p)}\mathbb{E}\Big[\eta([0,t_1]\times\cdots\times[0,t_p])^m\Big]dt_1\cdots dt_p\\
&\le\mathbb{E}\Big[\eta([0,1]^p)^m\Big]\int_0^\infty\cdots\int_0^\infty(t_1\cdots t_p)^{\frac{2p-d}{2p}m}e^{-(t_1+\cdots+t_p)}dt_1\cdots dt_p\\
&=\mathbb{E}\Big[\eta([0,1]^p)^m\Big]\Big[\Gamma\Big(\frac{2p-d}{2p}m+1\Big)\Big]^p.
\end{aligned}$$

By (3.16) and Stirling formula,

$$\liminf_{m\to\infty}\frac{1}{m}\log(m!)^{-d/2}\mathbb{E}\Big[\eta([0,1]^p)^m\Big]\ge\log\Big(\frac{2p}{\alpha p-d}\Big)^{\frac{2p-d}{2}}+\log\frac{\rho_1}{(2\pi)^d}. \tag{3.17}$$

On the other hand, notice that $\bar\tau\equiv\min\{\tau_1,\cdots,\tau_p\}$ has the exponential distribution with the parameter p. Hence,

$$\mathbb{E}\Big[\eta\big([0,\tau_1]\times\cdots\times[0,\tau_p]\big)^m\Big]\ge\mathbb{E}\Big[\eta\big([0,\bar\tau]^p\big)^m\Big]=\mathbb{E}\,\bar\tau^{\frac{2p-d}{2}m}\mathbb{E}\Big[\eta\big([0,1]^p\big)^m\Big]$$
$$=p^{-\frac{2p-d}{2}m-1}\Gamma\Big(1+\frac{2p-d}{2}m\Big)\mathbb{E}\Big[\eta\big([0,1]^p\big)^m\Big]$$

where the second step follows from (3.7). By Stirling formula and (3.16) again we have

$$\limsup_{m\to\infty}\frac{1}{m}\log(m!)^{-d/2}\mathbb{E}\Big[\eta([0,1]^p)^m\Big]\le\log\Big(\frac{2p}{2p-d}\Big)^{\frac{2p-d}{2}}+\log\frac{\rho_1}{(2\pi)^d}. \tag{3.18}$$

Combining (3.17)) and (3.18)) gives (3.13). □

4. Self-intersection of single Brownian path.

First we consider the case $d=1$. By (1.2) the self-intersection local time is essentially defined by the quantity

$$\int_{\mathbb{R}}L^p(t,x)dx. \tag{4.1}$$

The tail probability of the above quantity was computed in Mansmann (1991) in the special case $p=2$. He essentially proved that

$$\lim_{t\to\infty}t^{-2}\log\mathbb{P}\Big\{\int_{\mathbb{R}}L^2(1,x)dx\ge t\Big\}=-\frac{3}{2}.$$

We consider the general case in the following discussion. The following nice property of this model have been exploit extensively in Chen and Li (2004) and in Chen, Li and Rosen (2005): For any $s,t>0$, the quantity

$$\int_{\mathbb{R}}\big[L(s+t,x)-L(s,x)\big]^p dx$$

is independent of the family $\{W(u);\ u\le s\}$ and equal in law to the quantity given in (4.1).

By this property and by triangular inequality, one can prove that

$$\mathbb{E}\exp\left\{\theta\Big(\int_{\mathbb{R}} L^p(t,x)dx\Big)^{1/p}\right\} < \infty \qquad \forall \theta, t > 0.$$

Further, for any $s, t > 0$,

$$\begin{aligned}
&\mathbb{E}\exp\left\{\theta\Big(\int_{\mathbb{R}} L^p(s+t,x)dx\Big)^{1/p}\right\} \\
&\le \mathbb{E}\exp\left\{\theta\Big(\int_{\mathbb{R}} L^p(s,x)dx\Big)^{1/p}\right\}\mathbb{E}\exp\left\{\theta\Big(\Big(\int_{\mathbb{R}} L^p(t,x)dx\Big)^{1/p}\right\}.
\end{aligned}$$

By standard sub-additivity argument, the limit

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left\{\theta\Big(\int_{\mathbb{R}} L^p(t,x)dx\Big)^{1/p}\right\}$$

exists for any $\theta > 0$.

This observation gives, in comparison with (2.4), a strong indication as what to expect. Indeed, by a approximation via (2.4), or by applying the argument for (2.4) to the local time $L(t,x)$ instead of $L(t,x,\epsilon)$, we can show that the above limit is equal to

$$\begin{aligned}
&\sup_{f\in\mathcal{F}_1}\left\{\theta\Big(\int_{\mathbb{R}} |f(x)|^{2p}dx\Big)^{1/p} - \frac{1}{2}\int_{\mathbb{R}} |f'(x)|^2 dx\right\} \\
&= \theta^{\frac{2p}{p+1}}\sup_{f\in\mathcal{F}_1}\left\{\Big(\int_{\mathbb{R}} |f(x)|^{2p}dx\Big)^{1/p} - \frac{1}{2}\int_{\mathbb{R}} |f'(x)|^2 dx\right\}.
\end{aligned}$$

According to the identity given in Lemma 7.2 of Chen and Li (2004),

$$\begin{aligned}
&\sup_{f\in\mathcal{F}_1}\left\{\Big(\int_{\mathbb{R}} |f(x)|^{2p}dx\Big)^{1/p} - \frac{1}{2}\int_{\mathbb{R}} |f'(x)|^2 dx\right\} \\
&= p^{-\frac{2p}{p+1}}\left(\frac{\sqrt{2}}{(p-1)(p+1)}B\Big(\frac{1}{p-1},\frac{1}{2}\Big)\right)^{\frac{2(p-1)}{p+1}}
\end{aligned}$$

where $B(\cdot,\cdot)$ is beta function. By the scaling property

$$\int_{\mathbb{R}} L^p(t,x)dx \overset{d}{=} t^{\frac{p+1}{2}}\int_{\mathbb{R}} L^p(1,x)dx$$

and by Gärtner-Ellis theorem, we obtain the following result (Chen and Li (2004)).

Theorem 4.1. *Let $L(t,x)$ be the local time of an 1-dimensional Brownian motion. Then for any $p\ge 2$*

$$\lim_{t\to\infty} t^{\frac{2}{p-1}}\log\mathbb{P}\Big\{\int_{\mathbb{R}} L^p(1,x)dx\ge t\Big\} = -\frac{1}{4(p-1)}\Big(\frac{p+1}{2}\Big)^{\frac{3-p}{p-1}} B\Big(\frac{1}{p-1},\frac{1}{2}\Big)^2. \qquad (4.2)$$

We mention the fact that before Theorem 4.1 was obtained, Csörgö, Shi and Yor (1999) had established the exactly same form of large deviation for the local time of 1-dimensional Brownian bridge. Their argument is essentially based on excursion analysis and some distributional identities.

The self-intersection local time given in (1.1) can not be properly defines as $d\ge 2$. Take $p=2$ for example. A simple way to see this is to do the following symbolic calculation:

$$\mathbb{E}\,\beta([0,t]^2_<) = \iint_{\{0\le r<s\le t\}} \mathbb{E}\,\delta_0\big(W(r)-W(s)\big)drds = \iint_{\{0\le r<s\le t\}} \frac{1}{2\pi(s-r)}drds = \infty.$$

It suggests that the quantity $\beta([0,t]^2_<)$ blows up due to the contribution from the region close to the diagonal $\{(r,s)|\ r=s\}$. In other words, the problem is caused by the unbalance between short-range intersection and long-range intersection in the case $d\ge 2$ in which the short range self-intersection is too strong compared with long-range intersection.

This problem can be fixed for $d=2$ by renormalization. In this following discussion, we let $d=p=2$. Based on above observation, one can construct (by the usual way of approximation) the quantity

$$\beta(B) = \iint_B \delta_0\big(W(r)-W(s)\big)drds$$

if $B\subset\{(s,t)|\ r\le s\}$ is substantially away from the diagonal. In particular, if $B=[a,b]\times[b,c]$ for $0\le a<b<c$

$$\begin{aligned}\beta\big([a,b]\times[b,c]\big) &= \int_a^b\int_b^c \delta_0\Big(\big(W(r)-W(b)\big)-\big(W(s)-W(b)\big)\Big)dsdr\\ &= \int_0^{b-a}\int_0^{c-b} \delta_0\Big(\big(W(b-r)-W(b)\big)-\big(W(b+s)-W(b)\big)\Big)dsdr\\ &= \int_0^{b-a}\int_0^{c-b} \delta_0\big(W_1(r)-W_2(s)\big)dsdr\end{aligned}$$

where

$$W_1(r) = -\Big(W(b)-W(b-r)\Big)\ \ (0\le r\le b-a)\ \ \text{and}\ \ W_2(s)=W(b+s)-W(b)$$

are independent Brownian motions. Therefore, we have proved that

$$\beta\big([a,b]\times[b,c]\big)\overset{d}{=}\alpha\big([0,b-a]\times[0,c-b]\big) \tag{4.3}$$

where $\alpha([0,s]\times[0,t])$ is the intersection local time of two independent planar Brownian motions discussed in section 3.

We now construct the *renormalized self-intersection local time* formally given as

$$\begin{aligned}\gamma\big([0,t]^2_<\big) &= \iint_{\{0\le r<s\le t\}}\delta_0\big(W(r)-W(s)\big)drds\\ &\quad -\mathbb{E}\iint_{\{0\le r<s\le t\}}\delta_0\big(W(r)-W(s)\big)drds.\end{aligned}$$

The renormalization procedure we give in the following essentially belongs to Varadhan (1969). Let $t>0$ be fixed and set

$$A_l^k=\Big[\frac{2l}{2^{k+1}}t,\frac{2l+1}{2^{k+1}}t\Big)\times\Big(\frac{2l+1}{2^{k+1}}t,\frac{2l+2}{2^{k+1}}t\Big]\qquad l=0,1,\cdots,2^k-1,\quad k=0,1,\cdots.$$

An important fact is that the family

$$\Big\{A_l^k;\quad l=0,1,\cdots,2^k-1,\quad k=0,1,\cdots\Big\}$$

forms a partition of the triangular region $\{(r,s)|\ 0\le r<s\le t\}$. It will be reasonable to define

$$\gamma\big([0,t]^2_<\big)=\sum_{k=0}^\infty\bigg\{\sum_{l=0}^{2^k-1}\Big(\beta(A_l^k)-\mathbb{E}\,\beta(A_l^k)\Big)\bigg\} \tag{4.4}$$

if the right hand side converges in proper sense. Indeed, we have

Theorem 4.2. *The series*

$$\sum_{k=0}^\infty\bigg\{\sum_{l=0}^{2^k-1}\Big(\beta(A_l^k)-\mathbb{E}\,\beta(A_l^k)\Big)\bigg\}$$

convergences a.s. and in L^2.

Proof. The proof we give here comes from the monograph Le Gall (1992). By (4.3) we have that

$$\beta(A_l^k)\overset{d}{=}\alpha\big([0,2^{-(k+1)}]^2\big)\qquad l=0,1,\cdots,2^k-1,\quad k=0,1,\cdots.$$

In addition, for each $k\ge 0$, the finite sequence

$$\beta(A_l^k),\qquad l=0,1,\cdots,2^k-1$$

is independent. Consequently, by the scaling property (3.2) (with $d=p=2$)

$$\mathrm{Var}\Big(\sum_{l=0}^{2^k-1}\beta(A_l^k)\Big)=2^k\mathrm{Var}\Big(\alpha\big([0,2^{-(k+1)}]^2\big)\Big)=2^{-k-2}\mathrm{Var}\Big(\alpha\big([0,1]^2\big)\Big)=C2^{-k}.$$

Finally, the conclusion follows from the estimate

$$\bigg\{\mathbb{E}\bigg[\sum_{k=0}^\infty\bigg|\sum_{l=0}^{2^k-1}\Big(\beta(A_l^k)-\mathbb{E}\,\beta(A_l^k)\Big)\bigg|\bigg]^2\bigg\}^{1/2}\le\sum_{k=0}^\infty\bigg\{\mathrm{Var}\Big(\sum_{l=0}^{2^k-1}\beta(A_l^k)\Big)\bigg\}^{1/2}<\infty.$$

□

Now $\gamma\big([0,t]_<^2\big)$ is defined by (4.4) and is called renormalized self-intersection local time. It can be proved that for any $t>0$,

$$\gamma\big([0,t]_<^2\big)\overset{d}{=}t\gamma\big([0,1]_<^2\big).\tag{4.5}$$

Returning the polymer models discussed in (1.9) and (1.10). To remedy the problem caused by the ill-definition of $\beta\big([0,1]_<^2\big)$ we replace $\beta\big([0,1]_<^2\big)$ by $\gamma\big([0,1]_<\big)$. That is, we intend to redefine

$$\widehat{P}_\lambda(A)=\widehat{C}_\lambda^{-1}\mathbb{E}\exp\Big\{\lambda\gamma\big([0,1]_<^2\big)\Big\}1_{\{W(\cdot)\in A\}}\tag{4.6}$$

and

$$\widetilde{P}_\lambda(A)=\widetilde{C}_\lambda^{-1}\mathbb{E}\exp\Big\{-\lambda\gamma\big([0,1]_<^2\big)\Big\}1_{\{W(\cdot)\in A\}}\tag{4.7}$$

as the distributions of self-attraction polymer and self-repelling polymer, respectively. Beyond mathematical technicality, nothing significant has been changed— all we do is to renormalize model by the “constant” multiplier

$$\exp\Big\{\pm\lambda\mathbb{E}\,\beta\big([0,1]_<^2\big)\Big\}.$$

Nevertheless, this idea works only if

$$\mathbb{E}\exp\Big\{\lambda\gamma\big([0,1]_<^2\big)\Big\}<\infty\tag{4.8}$$

at least for λ in a neighborhood of 0. This fact was established by Le Gall (1994).

Theorem 4.3. *There is a $\lambda_0>0$ such that*

$$\mathbb{E}\exp\Big\{\lambda\gamma\big([0,1]_<^2\big)\Big\}\begin{cases}<\infty & \lambda<\lambda_0\\ =\infty & \lambda>\lambda_0.\end{cases}\tag{4.9}$$

In particular,

$$\mathbb{E}\exp\Big\{\lambda\gamma\big([0,1]^2_<\big)\Big\}<\infty \qquad \forall\lambda<0. \tag{4.10}$$

Proof. Take $t=1$ in the definition of A_l^k and write $\gamma(A)=\beta(A)-\mathbb{E}\,\beta(A)$. By Theorem 3.1 (with $d=p=2$) there is a $\lambda_1>0$ such that

$$\mathbb{E}\exp\Big\{\lambda_1\alpha\big([0,1]^2\big)\Big\}<\infty.$$

By Taylor expansion, therefore, there is a $C>0$ such that

$$\mathbb{E}\exp\Big\{\lambda\Big(\alpha\big([0,1]^2\big)-\mathbb{E}\,\alpha\big([0,1]^2\big)\Big)\Big\}\le e^{C\lambda^2} \qquad \lambda\le\lambda_1. \tag{4.11}$$

Fix $a\in(0,1)$. For each $N\ge 1$ set

$$\theta_N=2\lambda_1\prod_{j=2}^N\Big(1-2^{-a(j-1)}\Big)$$

($\theta_1=2\lambda_1$). By Hölder inequality

$$\begin{aligned}
&\mathbb{E}\exp\Big\{\theta_N\sum_{k=0}^N\sum_{l=0}^{2^k-1}\Big(\beta(A_l^k)-\mathbb{E}\,\beta(A_l^k)\Big)\Big\}\\
&\le\bigg[\mathbb{E}\exp\Big\{\theta_{N-1}\sum_{k=0}^{N-1}\sum_{l=0}^{2^k-1}\Big(\beta(A_l^k)-\mathbb{E}\,\beta(A_l^k)\Big)\Big\}\bigg]^{1-2^{-a(N-1)}}\\
&\times\bigg[\mathbb{E}\exp\Big\{2^{a(N-1)}\theta_N\sum_{l=0}^{2^N-1}\Big(\beta(A_l^N)-\mathbb{E}\,\beta(A_l^N)\Big)\Big\}\bigg]^{2^{-a(N-1)}}\\
&\le\mathbb{E}\exp\Big\{\theta_{N-1}\sum_{k=0}^{N-1}\sum_{l=0}^{2^k-1}\Big(\beta(A_l^k)-\mathbb{E}\,\beta(A_l^k)\Big)\Big\}\\
&\times\bigg[\mathbb{E}\exp\Big\{2^{a(N-1)}\theta_N\Big(\beta(A_0^N)-\mathbb{E}\,\beta(A_0^N)\Big)\Big\}\bigg]^{2^{(1-a)(N-1)}}.
\end{aligned}$$

By the fact that $\beta(A_0^N)\stackrel{d}{=}2^{-(N+1)}\alpha\big([0,1]^2\big)$ and by (4.11)

$$\begin{aligned}
&\mathbb{E}\exp\Big\{2^{a(N-1)}\theta_N\Big(\beta(A_0^N)-\mathbb{E}\,\beta(A_0^N)\Big)\Big\}\\
&=\mathbb{E}\exp\Big\{2^{-N-1+a(N-1)}\theta_N\Big(\alpha\big([0,1]^2\big)-\mathbb{E}\,\alpha\big([0,1]^2\big)\Big)\Big\}\\
&\le\exp\Big\{C\theta_N^2 2^{-2N+2a(N-1)}\Big\}.
\end{aligned}$$

Notice that $\theta_N \le 2\lambda_1$. Summarizing what we have

$$\begin{aligned}
&\mathbb{E}\,\exp\left\{\theta_N \sum_{k=0}^{N}\sum_{l=0}^{2^k-1}\Big(\beta(A_l^k) - \mathbb{E}\,\beta(A_l^k)\Big)\right\}\\
&\le \mathbb{E}\,\exp\left\{\theta_{N-1} \sum_{k=0}^{N-1}\sum_{l=0}^{2^k-1}\Big(\beta(A_l^k) - \mathbb{E}\,\beta(A_l^k)\Big)\right\}\exp\Big\{C' 2^{(a-1)N}\Big\}.
\end{aligned}$$

Repeating above procedure gives

$$\begin{aligned}
&\mathbb{E}\,\exp\left\{\theta_N \sum_{k=0}^{N}\sum_{l=0}^{2^k-1}\Big(\beta(A_l^k) - \mathbb{E}\,\beta(A_l^k)\Big)\right\}\\
&\le \exp\left\{C' \sum_{k=0}^{N} 2^{(a-1)k}\right\} \le \exp\left\{C'\Big(1-2^{a-1}\Big)^{-1}\right\} < \infty.
\end{aligned}$$

Notice that

$$\theta_\infty = 2\lambda_1 \prod_{j=2}^{\infty}\Big(1 - 2^{-a(j-1)}\Big) > 0.$$

By Fatou lemma we have

$$\mathbb{E}\,\exp\Big\{\theta_\infty \gamma\big([0,1]_<^2\big)\Big\} \le \exp\left\{C'\Big(1-2^{a-1}\Big)^{-1}\right\} < \infty.$$

On the other hand, notice that Theorem 3.2 also implies that there is a $\lambda_2 > 0$ such that

$$\mathbb{E}\,\exp\Big\{\lambda_2 \alpha\big([0,1]^2\big)\Big\} = \infty.$$

By a comparison argument one can prove that

$$\mathbb{E}\,\exp\Big\{\lambda \gamma\big([0,1]_<^2\big)\Big\} = \infty$$

if $\lambda > 0$ is large enough. □

In physics, the critical exponent $\lambda_0 > 0$ given in Theorem 4.3 represents the critical temperature between two different states of polymers. There are some concerns from physicists about the value of λ_0. From mathematical side, this problem is related to the tail of $\gamma\big([0,1]_<^2\big)$. The following result was given in Chen and Bass (2004).

Theorem 4.4.

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}\Big\{\gamma\big([0,1]_<^2\big) \ge t\Big\} = -\kappa(2,2)^{-4} \tag{4.12}$$

where $\kappa(2,2) > 0$ is the best constant of Gagliardo-Nirenberg inequality given in (3.4).

In addition, there is a constant $0 < L < \infty$ *such that for any* $\theta > 0$

$$\lim_{t\to\infty} t^{-2\pi\theta}\log\mathbb{P}\Big\{-\gamma\big([0,1]^2_<\big)\ge\theta\log t\Big\} = -L. \tag{4.13}$$

The upper tail behavior stated in (4.12) implies that $\lambda_0 = \kappa(2,2)^{-4}$. By the numerical approximation given in (3.6)

$$\lambda_0 \approx \pi\times 1.86225\cdots \approx 5.85043\cdots.$$

This is very close to a conjecture made by Duplantier (private communication).

Another interesting point about Theorem 4.4 is that despite that $\mathbb{E}\,\gamma\big([0,1]^2_<\big) = 0$, $\gamma\big([0,1]^2_<\big)$ has non-symmetric tails. Indeed, the lower tail given in (4.13) is much thinner than the upper tail given in (4.12).

The main ingredient in the proof for the upper tail (4.12) is to compare $\gamma\big([0,1]^2_<\big)$ with $\alpha\big([0,1]^2\big)$, the mutual intersection local time of two independent planar Brownian motions. To illustrate the idea, we only prove the upper bound. Consider the decomposition

$$\begin{aligned}\gamma\big([0,1]^2_<\big) &= \gamma\big([0,1/2]^2_<\big)+\gamma\big([1/2,1]^2_<\big)\\ &+\beta\big([0,1/2]\times[1/2,1]\big)-\mathbb{E}\,\beta\big([0,1/2]\times[1/2,1]\big).\end{aligned}$$

We have that $\gamma\big([0,1/2]^2_<\big)$ and $\gamma\big([1/2,1]^2_<\big)$ are independent and have the common distribution same as $(1/2)\gamma\big([0,1]^2_<\big)$. By (4.3),

$$\beta\big([0,1/2]\times[1/2,1]\big)\stackrel{d}{=}\alpha\big([0,1/2]^2\big)\stackrel{d}{=}\frac{1}{2}\alpha\big([0,1]^2\big). \tag{4.14}$$

Given $\epsilon>0$, by triangular inequality

$$\begin{aligned}&\mathbb{P}\Big\{\gamma\big([0,1]^2_<\big)\ge t\Big\}\\ &\le \mathbb{P}\Big\{\gamma\big([0,1/2]^2_<\big)+\gamma\big([1/2,1]^2_<\big)\ge\frac{1+\epsilon}{2}t\Big\}\\ &+\mathbb{P}\Big\{\beta\big([0,1/2]\times[1/2,1]\big)-\mathbb{E}\,\beta\big([0,1/2]\times[1/2,1]\big)\ge\frac{1-\epsilon}{2}t\Big\}.\end{aligned}$$

Consequently,

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t}\log P\Big\{\gamma\big([0,1]^2_<\big)\ge t\Big\}\\ &\le\max\Bigg\{\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}\Big\{\gamma\big([0,1/2]^2_<\big)+\gamma\big([1/2,1]^2_<\big)\ge\frac{1+\epsilon}{2}t\Big\},\\ &\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}\Big\{\beta\big([0,1/2]\times[1/2,1]\big)-\mathbb{E}\,\beta\big([0,1/2]\times[1/2,1]\big)\ge\frac{1-\epsilon}{2}t\Big\}\Bigg\}.\end{aligned}$$

By (4.14) and Theorem 3.1 (with $d = p = 2$) we have that the second limsup on the right hand side is equal to $-(1-\epsilon)\kappa(2,2)^{-4}$.

Let $\lambda_0 > 0$ be the critical exponent given in Theorem 4.3. We have

$$\limsup_{t\to\infty} \frac{1}{t} \log P\Big\{\gamma\big([0,1]^2_<\big) \ge t\Big\} = -\lambda_0. \tag{4.15}$$

Notice that

$$\mathbb{E}\exp\Big\{\lambda\Big(\gamma\big([0,1/2]^2_<\big) + \gamma\big([1/2,1]^2_<\big)\Big)\Big\} = \Big[\mathbb{E}\exp\Big\{\frac{\lambda}{2}\gamma\big([0,1]^2\big)\Big\}\Big]^2 < \infty$$

for every $\lambda < 2\lambda_0$. By Chebyshev inequality, this gives

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big\{\gamma\big([0,1/2]^2_<\big) + \gamma\big([1/2,1]^2_<\big) \ge \frac{1+\epsilon}{2} t\Big\} \le -(1+\epsilon)\lambda_0.$$

Summarizing our argument, we have obtained

$$-\lambda_0 \le \max\Big\{ -(1+\epsilon)\lambda_0, \quad -(1-\epsilon)\kappa(2,2)^{-4}\Big\}.$$

So we must have $\lambda_0 \ge (1-\epsilon)\kappa(2,2)^{-4}$. Letting $\epsilon \to 0$ we have $\lambda_0 \ge \kappa(2,2)^{-4}$. Thus the upper bound of (4.12) follows from (4.15).

The idea for the lower tail (4.13) is sub-additivity. One needs only to prove (4.13) in the case $\theta = (2\pi)^{-1}$. That is,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big\{ -\gamma\big([0,1]^2_<\big) \ge (2\pi)^{-1}\log t\Big\} = -L. \tag{4.16}$$

Indeed, the general statement will follow if we substitute t by $t^{2\pi\theta}$.

Given $s, t > 0$,

$$\begin{aligned}
&\gamma\big([0,s+t]^2_<\big) = \gamma\big([0,s]^2_<\big) + \gamma\big([s,s+t]^2_<\big) \\
&+ \beta\big([0,s]\times[s,s+t]\big) - \mathbb{E}\,\beta\big([0,s]\times[s,s+t]\big) \\
&\ge \gamma\big([0,s]^2_<\big) + \gamma\big([s,s+t]^2_<\big) - \mathbb{E}\,\beta\big([0,s]\times[s,s+t]\big).
\end{aligned}$$

By a simple computation,

$$\begin{aligned}
&\mathbb{E}\,\beta\big([0,s]\times[s,s+t]\big) = \int_0^s \int_s^{s+t} \frac{1}{2\pi}\frac{1}{v-u} dv du \\
&= \frac{1}{2\pi}\big[(s+t)\log(s+t) - s\log s - t\log t\big].
\end{aligned}$$

By independence between $\gamma\big([0,s]^2_<\big)$ and $\gamma\big([s,s+t]^2_<\big)$, therefore,

$$
\begin{aligned}
&(s+t)^{-(s+t)}\mathbb{E}\,\exp\Big\{-2\pi\gamma\big([0,s+t]^2_<\big)\Big\}\\
&\le \bigg(s^{-s}\mathbb{E}\,\exp\Big\{-2\pi\gamma\big([0,s]^2_<\big)\Big\}\bigg)\bigg(t^{-t}\mathbb{E}\,\exp\Big\{-2\pi\gamma\big([0,t]^2_<\big)\Big\}\bigg)
\end{aligned}
$$

or, $a(s+t)\le a(s)+a(t)$, where

$$
a(t)=\log\bigg(t^{-t}\mathbb{E}\,\exp\Big\{-2\pi\gamma\big([0,t]^2_<\big)\Big\}\bigg).
$$

This sub-additivity property implies that the limit

$$
\lim_{t\to\infty}\frac{a(t)}{t}=\inf_{t>0}\Big\{\frac{a(t)}{t}\Big\}
$$

exists and is finite. Let $t=n$ be integer. By the scaling $\gamma\big([0,n]^2_<\big)\overset{d}{=}n\gamma\big([0,1]^2_<\big)$ and by Stirling formula,

$$
\lim_{n\to\infty}\frac{1}{n}\log\bigg(\frac{1}{n!}\mathbb{E}\,\exp\Big\{-2\pi n\gamma\big([0,1]^2_<\big)\Big\}\bigg)=1+\inf_{t>0}\Big\{\frac{a(t)}{t}\Big\}.
$$

Applying Lemma 3.1 to the non-negative random variable

$$
Y=\exp\Big\{-2\pi\gamma\big([0,1]^2_<\big)\Big\}
$$

gives (4.16) with

$$
L=\exp\bigg\{-1-\inf_{t>0}\Big\{\frac{a(t)}{t}\Big\}\bigg\}>0.
$$

A minorization estimate of the probability

$$
\mathbb{P}\Big\{-\gamma\big([0,1]^2_<\big)\ge(2\pi)^{-1}\log t\Big\}
$$

gives that $L<\infty$. $\square$

Most of the results that we have stated by far have been extended from the setting of Brownian motions to the setting of symmetric stable processes. We refer reader to Bass, Chen and Rosen (2005), Chen, Li and Rosen (2005), Chen and Rosen (2005) for these extensions. Sometimes, the stable case presents some unusual patterns which are sharply contrary to those found in the Brownian case. Take the lower tail behavior of the renormalized 2-multiple self-intersection local time as an example. Given a d-dimensional stable process with index $\alpha\in(0,2]$, its 2-multiple self-intersection local time has to be renormalized and can be renormalized if and only if $2d/3 < \alpha \le d$. As $d = \alpha$, which corresponds to 2-dimensional Brownian case and 1-dimensional Cauchy case, it has been found that the lower tail of the renormalized self-intersection local time $\gamma\big([0,t]^2_{\le}\big)$ falls into the pattern described by (4.13). In the non-critical cases defined by $2d/3 < \alpha < d$, on the other hand, it was proved in Bass, Chen and Rosen (2005) that there is a $0 < b < \infty$, such that

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}\Big\{-\gamma\big([0,1]^2_{\le}\big)\ge t^{d/\alpha-1}\Big\} = -b.$$

The problem on the tail probabilities for renormalized self-intersection local times with intersection multiple $p \ge 3$ remains open. Given a 2-dimensional Brownian motion $W(t)$, the renormalized p-multiple self intersection local time is formally defined (Rosen (1986)) as

$$\gamma\big([0,t]^p_<\big) = \int\cdots\int_{\{0\le s_1<\cdots<s_p\le t\}}\overline{\prod_{j=2}^p\delta_0\big(W(s_j)-W(s_{j-1})\big)}ds_1\cdots ds_p$$

where for any random variable X, we use the notation $\overline{X}$ for its centralization $X - \mathbb{E}\,X$. It can be shown that

$$\gamma\big([0,t]^p_<\big) \overset{d}{=} t\gamma\big([0,1]^p_<\big).$$

To the best of the author's knowledge, nothing has been known about the tail behaviors of $\gamma\big([0,t]^p_<\big)$ in the case $p \ge 3$, to which the argument used in Theorem 4.3 and Theorem 4.4 is no longer applicable. Here we conjecture that there is a $\lambda_0 > 0$ such that

$$\mathbb{E}\exp\Big\{\lambda\big|\gamma\big([0,1]^p_<\big)\big|^{\frac{1}{p-1}}\Big\}\begin{cases} <\infty & \lambda < \lambda_0 \\ =\infty & \lambda > \lambda_0. \end{cases}$$

We also ask for the identification of the critical value λ_0.

In the case $d \ge 3$, the self-intersection local time of a d-dimensional Brownian motion can not be renormalized. The problem is to study the limiting behavior of the quantity

$$\beta_\epsilon\big([0,t]^2_<\big) = \iint_{0\le r<s\le t} f_\epsilon\big(W(s)-W(r)\big)drds$$

as $\epsilon \to 0^+$, where $f(x)$ is a nice density function on $\mathbb{R}^d$ and $f_\epsilon(x) = \epsilon^{-d}f(\epsilon^{-1}x)$. Yor (1985) and Calais and Yor (1987) have proved that random variables

$$\begin{cases} \big(\log(1/\epsilon)\big)^{-1/2}\Big\{\beta_\epsilon\big([0,1]^2_<\big) - \mathbb{E}\,\beta_\epsilon\big([0,1]^2_<\big)\Big\} & \text{if } d = 3 \\ \epsilon^{\frac{d-3}{2}}\Big\{\beta_\epsilon\big([0,1]^2_<\big) - \mathbb{E}\,\beta_\epsilon\big([0,1]^2_<\big)\Big\} & \text{if } d \ge 4 \end{cases}$$

weakly converge to symmetric normal distributions. Consequently, a Gaussian tail is expected for $\beta_\epsilon\big([0,1]^2_<\big) - \mathbb{E}\,\beta_\epsilon\big([0,1]^2_<\big)$. The reader is referred to section 6 for its implementation on the tail behaviors for self-intersection local times and ranges of random walks with dimension $d \geq 3$.

5. Intersection of independent random walks.

Let $\{S_1(n)\}, \cdots, \{S_p(n)\}$ be independent symmetric random walks on $\mathbb{Z}^d$. Through this section, we assume that they have same distribution and that their smallest supporting group is $\mathbb{Z}^d$. Let Γ be their covariance matrix.

It is well known that

$$\#\big\{S_1(0,\infty)\cap\cdots\cap S_p(0,\infty)\big\} = \infty \qquad a.s.$$

if any only if $p(d-2) \leq d$. where for given $A \subset \mathbb{Z}^+$, $S_j(A) = \{S_j(k);\ k \in A\}$.

Under $p(d-2) \leq d$, we are interested in the asymptotic behaviors of the intersection local time I_n and the intersection J_n of the independent ranges, where I_n and J_n are defined by (1.5) and (1.7), respectively. I_n counts the times that the random walks intersect in n steps while J_n counts the sites where the intersection takes place. An easy observation is that $J_n \leq I_n$, where the inequality is caused by the possibility that intersection takes place at same site more than once. Since the random walks in high dimension ($d \geq 3$) can only intersect finitely many times at the same site, we expect that I_n and J_n differ only by a constant multiplier in this case.

In the case $p(d-2) > d$,

$$I_\infty = \#\Big\{(k_1, \cdots, k_p) \in (0,\infty)^d;\quad S_1(k_1) = \cdots = S_p(k_p)\Big\} < \infty \tag{5.1}$$

$$J_\infty = \#\big\{S_1(0,n]\cap\cdots\cap S_p(0,n]\big\} < \infty \tag{5.2}$$

with probability 1. The question is on the tail probabilities of I_∞ and J_∞.

In the following discussion, we consider three very different situations: $p(d-2) < d$, $p(d-2) = d$ and $p(d-2) > d$, which are referred as sub-critical dimensions, critical dimensions and super-critical dimensions, respectively.

The critical dimensions consist of two cases: the case $d = 4$, $p = 2$ and the case $d = p = 3$. Le Gall (1986b) proved that as $d = 4$ and $p = 2$

$$(\log n)^{-1} I_n \xrightarrow{d} (2\pi)^{-2}\det(\Gamma)^{-1/2} U^2 \tag{5.3}$$

$$(\log n)^{-1} J_n \xrightarrow{d} \gamma_S^2 (2\pi)^{-2}\det(\Gamma)^{-1/2} U^2 \tag{5.4}$$

where U is a $N(0,1)$ random variable and

$$\gamma_S = \mathbb{P}\{S(n) \neq 0 \ \forall n \geq 1\}. \tag{5.5}$$

In the same paper, Le Gall also proved that when $d = p = 3$,

$$(\log n)^{-1} I_n \xrightarrow{d} (2\pi)^{-2} \frac{1}{\det(\Gamma)} V \tag{5.6}$$

$$(\log n)^{-1} J_n \xrightarrow{d} (2\pi)^{-2} \frac{\gamma_S^3}{\det(\Gamma)} V \tag{5.7}$$

where the random variable V has the gamma distribution with parameter $1/4$.

Therefore, it is reasonable to expect that I_n and J_n have gamma tails in the critical dimensions. This was confirmed in Marcus-Rosen (1997) and in Rosen (1997) in the form of law of the iterated logarithm.

Theorem 5.1. *Under $d = 4$ and $p = 2$,*

$$\limsup_{n\to\infty} (\log n)^{-1} (\log\log\log n)^{-1} I_n = (2\pi^2)^{-1} \det(\Gamma)^{-1/2} \quad a.s. \tag{5.8}$$

$$\limsup_{n\to\infty} (\log n)^{-1} (\log\log\log n)^{-1} J_n = \gamma_S^2 (2\pi^2)^{-1} \det(\Gamma)^{-1/2} \quad a.s. \tag{5.9}$$

Under $d = p = 3$,

$$\limsup_{n\to\infty} (\log n)^{-1} (\log\log\log n)^{-1} I_n = \frac{1}{\pi \det(\Gamma)} \quad a.s. \tag{5.10}$$

$$\limsup_{n\to\infty} (\log n)^{-1} (\log\log\log n)^{-1} J_n = \frac{\gamma_S^3}{\pi \det(\Gamma)} \quad a.s. \tag{5.11}$$

In the sub-critical dimensions $p(d-2) < d$, Le Gall (1986a) proved that

$$n^{-\frac{2p-d(p-1)}{2}} I_n \xrightarrow{d} \det(\Gamma)^{-\frac{p-1}{2}} \alpha\big([0,1]^p\big). \tag{5.12}$$

where $\alpha\big([0,1]^p\big)$ is the intersection local time of p independent d-dimensional Brownian motions.

The weak laws for J_n are more complicated and they further break the sub-critical dimensions into three cases: the case $d = 1$, $p \geq 2$; the case $d = 2$, $p \geq 2$; and the case $d = 3$, $p = 2$. The following results were established in Le Gall (1986a) and Rosen (1990):

As $d = 1$ and $p \geq 2$,

$$\frac{1}{\sqrt{n}} J_n \xrightarrow{d} \sigma\Big(\min_{1\leq j\leq p} \max_{1\leq t\leq 1} W_j(t) - \max_{1\leq j\leq p} \min_{1\leq t\leq 1} W_j(t) \Big) \tag{5.13}$$

where $W_1(t), \cdots, W_p(t)$ are independent 1-dimensional Brownian motions.

As $d = 2$ and $p \geq 2$,

$$\frac{(\log n)^p}{n} J_n \xrightarrow{d} (2\pi)^p \sqrt{\det(\Gamma)} \alpha\big([0,1]^p\big) \tag{5.14}$$

where $\alpha\big([0,1]^p\big)$ is intersection local time run by p independent planar Brownian motions.

As $d = 3$ and $p = 2$,

$$\frac{1}{\sqrt{n}} J_n \xrightarrow{d} \gamma_S^2 \det(\Gamma)^{-\frac{1}{2}} \alpha\big([0,1]^2\big) \tag{5.15}$$

where $\alpha\big([0,1]^p\big)$ is intersection local time between two 3-dimensional independent Brownian motions.

To investigate the tail probabilities of I_n and J_n in the sub-critical dimensions, we begin with the critical integrability of I_n and J_n. By Theorem 3.1, the critical integrability of $\alpha\big([0,1]^2\big)$ is described as following:

$$\mathbb{E} \exp\left\{\theta\Big(\alpha\big([0,1]^p\big)\Big)^{\frac{2}{d(p-1)}}\right\} \begin{cases} < \infty & \theta < \frac{p}{2}\kappa(d,p)^{-\frac{4p}{d(p-1)}} \\ = \infty & \theta > \frac{p}{2}\kappa(d,p)^{-\frac{4p}{d(p-1)}}. \end{cases} \tag{5.16}$$

The following theorem was given in Bass, Chen and Rosen (2006, 2007).

Theorem 5.2. *Under $p(d-2) < d$, there is a $\theta > 0$ such that*

$$\sup_{n\geq 1} \mathbb{E} \exp\left\{\theta n^{-\frac{2p-d(p-1)}{d(p-1)}} I_n^{\frac{2}{d(p-1)}}\right\} < \infty. \tag{5.17}$$

In the case $d = 1$ and $p \geq 2$,

$$\sup_{n\geq 1} \mathbb{E} \exp\left\{\theta n^{-1} J_n^2\right\} < \infty. \tag{5.18}$$

In the case $d = 2$ and $p \geq 2$,

$$\sup_{n\geq 1} \mathbb{E} \exp\left\{\theta \frac{(\log n)^p}{n} J_n\right\} < \infty. \tag{5.19}$$

In the case $d = 3$ and $p = 2$,

$$\sup_{n\geq 1} \mathbb{E} \exp\left\{\theta n^{-1/3} J_n^{2/3}\right\} < \infty. \tag{5.20}$$

The integrability established for I_n and J_n are best possible in the sense that there is a $\theta_0 > 0$ such that left hand sides of (5.17)—(5.20) become infinity if $\theta > \theta_0$. This can be seen from Le Gall's weak laws and from the critical integrability of $\alpha([0,1]^p)$ given in (5.16).

The approach for Theorem 5.2 relies on two steps of moment estimate. The moment estimate in the first step goes back at least to Le Gall and Rosen (1991). Write

$$I_n = \sum_{x\in\mathbb{Z}^d}\prod_{j=1}^p l_j(n,x) \quad\text{and}\quad J_n = \sum_{x\in\mathbb{Z}^d}\prod_{j=1}^p 1_{\{T_x^j\le n\}} \tag{5.21}$$

where $T_x^j = \inf\{k\ge 1;\ S_j(k)=x\}$. For each integer $m\ge 1$,

$$\begin{aligned}
\mathbb{E}\, I_n^m &= \sum_{x_1,\cdots,x_m\in\mathbb{Z}^d}\Big[\mathbb{E}\prod_{k=1}^m l(n,x_k)\Big]^p\\
&\le \sum_{x_1,\cdots,x_m\in\mathbb{Z}^d}\Big[\sum_{\sigma\in\Sigma_m}\sum_{1\le i_1\le\cdots\le i_m\le n}\mathbb{E}\prod_{k=1}^m 1_{\{S(i_k)=x_{\sigma(k)}\}}\Big]^p\\
&= \sum_{x_1,\cdots,x_m\in\mathbb{Z}^d}\Big[\sum_{\sigma\in\Sigma_m}\sum_{1\le i_1\le\cdots\le i_m\le n}\prod_{k=1}^m \mathbb{P}\{S(i_k-i_{k-1})=x_{\sigma(k)}-x_{\sigma(k-1)}\}\Big]^p
\end{aligned}$$

where Σ_m is the permutation group on $\{1,\cdots,m\}$. Notice that

$$\sum_{\sigma\in\Sigma_m}\sum_{1\le i_1\le\cdots\le i_m\le n}\prod_{k=1}^m \mathbb{P}\{S(i_k-i_{k-1})=x_{\sigma(k)}-x_{\sigma(k-1)}\}\le\prod_{k=1}^m\mathbb{E}\,\widetilde{l}(n,x_{\sigma(k)}-x_{\sigma(k-1)})$$

where $\widetilde{l}(n,x)$ is the augmented local time

$$\widetilde{l}(n,x)=\sum_{k=0}^n 1_{\{S(k)=x\}}.$$

Hence,

$$\begin{aligned}
\mathbb{E}\, I_n^m &\le \sum_{x_1,\cdots,x_m\in\mathbb{Z}^d}\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m\mathbb{E}\,\widetilde{l}(n,x_{\sigma(k)}-x_{\sigma(k-1)})\Big]^p\\
&\le (m!)^{p-1}\sum_{x_1,\cdots,x_m\in\mathbb{Z}^d}\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m\Big[\mathbb{E}\,\widetilde{l}(n,x_{\sigma(k)}-x_{\sigma(k-1)})\Big]^p \qquad (5.22)\\
&= (m!)^p\Big(\sum_{x\in\mathbb{Z}^d}\Big[\mathbb{E}\,\widetilde{l}(n,x)\Big]^p\Big)^m = (m!)^p\Big(\mathbb{E}\,\widetilde{I}_n\Big)^m
\end{aligned}$$

where $\widetilde{I}_n$ is the augmented intersection local time

$$\widetilde{I}_n = \#\Big\{(k_1, \cdots, k_p) \in [0, n]^p; \quad S_1(k_1) = \cdots = S_p(k_p)\Big\}.$$

Similarly,

$$\mathbb{E}\, J_n^m \le (m!)^p \Big(\mathbb{E}\, \widetilde{J}_n\Big)^m. \tag{5.23}$$

Unfortunately, (5.22) and (5.23) are not sharp enough for Theorem 5.3. We need to reduce the power on $m!$ by the second step of moment estimate. To this end we introduce the following moment inequalities originally given in Chen (2004, 2005).

Theorem 5.3. *For any integers $m \ge 1$, $a \ge 2$ and $n_1, \cdots, n_a \ge 1$*

$$\big(\mathbb{E}\, I^m_{n_1+\cdots+n_a}\big)^{1/p} \le \sum_{\substack{k_1+\cdots+k_a=m\\ k_1,\cdots,k_a\ge 0}} \frac{m!}{k_1!\cdots k_a!}\big(\mathbb{E}\, I^{k_1}_{n_1}\big)^{1/p}\cdots\big(\mathbb{E}\, I^{k_a}_{n_a}\big)^{1/p} \tag{5.24}$$

$$\big(\mathbb{E}\, J^m_{n_1+\cdots+n_a}\big)^{1/p} \le \sum_{\substack{k_1+\cdots+k_a=m\\ k_1,\cdots,k_a\ge 0}} \frac{m!}{k_1!\cdots k_a!}\big(\mathbb{E}\, J^{k_1}_{n_1}\big)^{1/p}\cdots\big(\mathbb{E}\, J^{k_a}_{n_a}\big)^{1/p}. \tag{5.25}$$

We improve the estimate (5.22) by showing that there is a constant $C > 0$ such that

$$\mathbb{E}\, I_n^m \le C^m (m!)^{\frac{d(p-1)}{2}} n^{\frac{2p-d(p-1)}{2}m} \qquad m, n = 1, 2, \cdots. \tag{5.26}$$

We first deal with the case $m \le n$. Take $a = m$ and $n_1 = \cdots = n_m = [n/m] + 1$ in (5.24). By (5.22) and by the fact $m \le n$,

$$\mathbb{E}\, I^{k_i}_{[n/m]+1} \le c_1^{k_i}(k_i!)^p\Big(\frac{n}{m}\Big)^{\frac{2p-d(p-1)}{2}k_i} \qquad i = 1, \cdots, m.$$

By (5.24),

$$\begin{aligned}\Big(\mathbb{E}\, I_n^m\Big)^{1/p} &\le \sum_{\substack{k_1+\cdots+k_m=m\\ k_1,\cdots,k_m\ge 0}} \frac{m!}{k_1!\cdots k_m!} c_1^{m/p} k_1!\cdots k_m!\Big(\frac{n}{m}\Big)^{\frac{2p-d(p-1)}{2p}m}\\ &= m! c_1^{m/p}(m^m)^{-\frac{2p-d(p-1)}{2p}} n^{\frac{2p-d(p-1)}{2p}m}\binom{2m-1}{m}.\end{aligned}$$

Therefore, (5.26) follows from the easy bounds $m^m \ge m!$ and

$$\binom{2m-1}{m} \le \sum_{k=0}^{2m-1}\binom{2m-1}{k} = 2^{2m-1}.$$

As for the case $m > n$, by the fact that $I_n \le n^p$ we have the trivial bound

$$\mathbb{E}\, I_n^m \le n^{pm} \le m^{\frac{d(p-1)}{2}m} n^{\frac{2p-d(p-1)}{2}m} \le C^m (m!)^{\frac{d(p-1)}{2}} n^{\frac{2p-d(p-1)}{2}m}$$

where the last step follows from Stirling formula.

Clearly, (5.26) implies (5.17) via Taylor expansion. The rest of Theorem 5.3 follows from the similar treatment applied to the moment of J_n. $\square$

By the fact that Theorem 5.2 gives the optional integrability to I_n and J_n. It is natural to believe that it provides sharp (up to constant) bounds for the tail probabilities of I_n and J_n, by a simple use of Chebyshev inequality. In the case of I_n, that means that the estimate

$$\mathbb{P}\Big\{ I_n \ge n^{\frac{2p-d(p-1)}{2}} b_n^{\frac{d(p-1)}{2}} \Big\} \le e^{-\theta b_n} \mathbb{E} \exp\Big\{ \theta n^{-\frac{2p-d(p-1)}{d(p-1)}} I_n^{\frac{2}{d(p-1)}} \Big\}$$

suggests that for suitable b_n, the tail probability

$$\mathbb{P}\Big\{ I_n \ge n^{\frac{2p-d(p-1)}{2}} b_n^{\frac{d(p-1)}{2}} \Big\}$$

has a decay rate $e^{-\theta b_n}$ for some $\theta > 0$. More precisely, we have (Chen (2004)) the following theorem.

Theorem 5.4. *Under $p(d-2) < d$,*

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\big\{ I_n \ge n^{\frac{2p-d(p-1)}{2}} b_n^{\frac{d(p-1)}{2}} \big\} = -\frac{p}{2} \det(\Gamma)^{1/d} \kappa(d,p)^{-\frac{4p}{d(p-1)}} \tag{5.27}$$

for any positive sequence $\{b_n\}$ satisfying

$$b_n \longrightarrow \infty \quad \text{and} \quad b_n = o(n) \qquad (n \to \infty)$$

where $\kappa(d,p)$ is the best constant of Gagliardo-Nirenberg inequality given in (3.4).

For the intersection of J_n of the independent ranges in sub-critical dimensions, we have (Chen (2005, 2006b)) the following theorem.

Theorem 5.5. *Let b_n be a positive sequence satisfying $b_n \to \infty$ and the restrictions in each of the following cases.*

(1). As $d = 1$ and $p \ge 2$,

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\big\{ J_n \ge \sqrt{n b_n} \big\} = -\frac{p}{2\sigma^2} \tag{5.28}$$

for $b_n = o(n)$, where σ^2 is the variance of the random walks.

(2). As $d = 2$ and $p \geq 2$,

$$\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\Big\{J_n \geq \frac{n}{(\log n)^p}b_n^{p-1}\Big\} = -\frac{p}{2}(2\pi)^{-\frac{p}{p-1}}\det(\Gamma)^{-\frac{1}{2(p-1)}}\kappa(2,p)^{-\frac{2p}{p-1}} \tag{5.29}$$

for $b_n = o\big((\log n)^{2/3}\big)$.

(3). As $d = 3$ and $p = 2$,

$$\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\Big\{J_n \geq \sqrt{nb_n^3}\Big\} = -\det(\Gamma)^{1/3}\gamma^{-4/3}\kappa(3,2)^{-8/3} \tag{5.30}$$

for $b_n = o\big(n^{2/9}\big)$.

By comparing Theorem 5.4 and 5.5 to Theorem 3.1, one can see how the weak laws pass the large deviations from $\alpha([0,1]^p)$ to I_n and J_n, except the case $d = 1$ from J_n (which is the discrete version of the tail for the intersection of the ranges of independent 1-dimensional Brownian motions). This fact is also reflected by the proof of Theorem 5.4 and 5.5. In the following we try to sketch some of the key ideas behind Theorem 5.4.

We first recall the following lemma, which was given in Chen (2005).

Lemma 5.1. *Let $\{Y_n\}$ be a sequence of non-negative random variables and let $\{b_n\}$ be a positive sequence with $b_n \to \infty$ such that for any $\theta > 0$,*

$$\lim_{n\to\infty}\frac{1}{b_n}\log\sum_{m=0}^{\infty}\frac{\theta^m}{m!}b_n^m\Big(\mathbb{E}\,Y_n^m\Big)^{1/p} = \Psi(\theta) \tag{5.31}$$

where $\Psi(\theta)$ is essentially smooth in its domain. Then

$$\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\big\{Y_n \geq \lambda\big\} = -I(\lambda) \qquad (\lambda > 0) \tag{5.32}$$

where

$$I(\lambda) = p\sup_{\theta>0}\Big\{\lambda^{1/p}\theta - \Psi(\theta)\Big\}.$$

To prove Theorem 5.4, we need only to verify that for any $\theta > 0$

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{b_n}\log\sum_{m=0}^{\infty}\frac{\theta^m}{m!}\Big(\frac{b_n}{n}\Big)^{\frac{2p-d(p-1)}{2p}m}\Big(\mathbb{E}\,I_n^m\Big)^{1/p}\\ &= 2^{\frac{d(p-1)}{2p-d(p-1)}}\frac{2p-d(p-1)}{2p}\Big(\frac{d(p-1)}{2p}\Big)^{\frac{d(p-1)}{2p-d(p-1)}}\\ &\times\det(\Gamma)^{-\frac{p-1}{2p-d(p-1)}}\kappa(d,p)^{\frac{4p}{2p-d(p-1)}}\theta^{\frac{2p}{2p-d(p-1)}}.\end{aligned} \tag{5.33}$$

Notice that (5.24) in Theorem 5.3 implies that for any $\theta > 0$,

$$\sum_{m=0}^{\infty} \frac{\theta^m}{m!}\Big(\mathbb{E}\, I^m_{n_1+\cdots+n_a}\Big)^{1/p} \le \prod_{i=1}^{a}\sum_{m=0}^{\infty} \frac{\theta^m}{m!}\Big(\mathbb{E}\, I^m_{n_i}\Big)^{1/p}.$$

Let $t > 0$ be fixed and let $t_n = [tn/b_n]$ and $\gamma_n = [n/t_n]$. Then $n \le t_n(\gamma_n + 1)$. By taking $a = \gamma_n + 1$,

$$\sum_{m=0}^{\infty} \frac{1}{m!}\theta^m\Big(\frac{b_n}{n}\Big)^{\frac{2p-d(p-1)}{2p}m}\big(\mathbb{E}\, I^m_n\big)^{1/p} \le \bigg(\sum_{m=0}^{\infty} \frac{1}{m!}\theta^m\Big(\frac{b_n}{n}\Big)^{\frac{2p-d(p-1)}{2p}m}\big(\mathbb{E}\, I^m_{t_n}\big)^{1/p}\bigg)^{\gamma_n+1}.$$

By the weak convergence given in (5.12) we have

$$\begin{aligned}&\sum_{m=0}^{\infty} \frac{1}{m!}\theta^m\Big(\frac{b_n}{n}\Big)^{\frac{2p-d(p-1)}{2p}m}\big(\mathbb{E}\, I^m_{t_n}\big)^{1/p}\\ &\longrightarrow \sum_{m=0}^{\infty} \frac{1}{m!}\theta^m t^{\frac{2p-d(p-1)}{2p}m}\det(\Gamma)^{-\frac{p-1}{2p}m}\big(\mathbb{E}\,\alpha\big([0,1]^p\big)^m\big)^{1/p}\end{aligned}$$

as $n \to \infty$. Hence,

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{b_n}\log\bigg(\sum_{m=0}^{\infty} \frac{1}{m!}\theta^m\Big(\frac{b_n}{n}\Big)^{\frac{2p-d(p-1)}{2p}m}\big(\mathbb{E}\, I^m_n\big)^{1/p}\bigg)\\ &\le \frac{1}{t}\log\bigg(\sum_{m=0}^{\infty} \frac{1}{m!}\theta^m t^{\frac{2p-d(p-1)}{2p}m}\det(\Gamma)^{-\frac{p-1}{2p}m}\Big(\mathbb{E}\,\alpha\big([0,1]^p\big)^m\Big)^{1/p}\bigg).\end{aligned}$$

By an inverse of the general result given in Lemma 5.1, Theorem 3.1 implies that

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t}\log\bigg(\sum_{m=0}^{\infty} \frac{1}{m!}\theta^m t^{\frac{2p-d(p-1)}{2p}m}\det(\Gamma)^{-\frac{p-1}{2p}m}\Big(\mathbb{E}\,\alpha\big([0,1]^p\big)^m\Big)^{1/p}\bigg)\\ &= 2^{\frac{d(p-1)}{2p-d(p-1)}}\,\frac{2p-d(p-1)}{2p}\Big(\frac{d(p-1)}{2p}\Big)^{\frac{d(p-1)}{2p-d(p-1)}}\\ &\times \det(\Gamma)^{-\frac{p-1}{2p-d(p-1)}}\kappa(d,p)^{\frac{4p}{2p-d(p-1)}}\theta^{\frac{2p}{2p-d(p-1)}}.\end{aligned}$$

Thus, we have established the upper bound for (5.33).

We now come to the lower bound of (5.33). Use $[x]$ for the lattice part of $x \in \mathbb{R}^d$. Notice that

$$I_n = \sum_{x\in\mathbb{Z}^d}\prod_{j=1}^{p} l_j(n,x) = \int_{\mathbb{R}^d}\prod_{j=1}^{p} l_j\big(n,[x]\big)dx = \Big(\frac{n}{b_n}\Big)^{d/2}\int_{\mathbb{R}^d}\prod_{j=1}^{p} l_j\big(n,[\sqrt{n/b_n}x]\big)dx.$$

For any integer $m \geq 1$,

$$\begin{aligned} \mathbb{E}\, I_n^m &= \Big(\frac{n}{b_n}\Big)^{\frac{md}{2}} \mathbb{E} \int_{(\mathbb{R}^d)^m} dx_1 \cdots dx_m \prod_{j=1}^p \prod_{k=1}^m l_j\big(n, [\sqrt{n/b_n} x_k]\big) \\ &= \Big(\frac{n}{b_n}\Big)^{\frac{md}{2}} \int_{(\mathbb{R}^d)^m} dx_1 \cdots dx_m \Big[\prod_{k=1}^m l\big(n, [\sqrt{n/b_n} x_k]\big)\Big]^p. \end{aligned}$$

Let $q > 1$ be given by $p^{-1} + q^{-1} = 1$ and let f be a nice function on $\mathbb{R}^d$ such that $||f||_q = 1$.

$$\begin{aligned} \big(\mathbb{E}\, I_n^m\big)^{1/p} &\geq \Big(\frac{n}{b_n}\Big)^{\frac{md}{2p}} \int_{(\mathbb{R}^d)^m} dx_1 \cdots dx_m \Big[\prod_{k=1}^m f(x_k)\Big]\Big[\prod_{k=1}^m l\big(n, [\sqrt{n/b_n} x_k]\big)\Big] \\ &= \Big(\frac{n}{b_n}\Big)^{\frac{md}{2p}} \mathbb{E}\Big[\int_{\mathbb{R}^d} f(x) l\big(n, [\sqrt{n/b_n} x]\big) dx\Big]^m \\ &= \Big(\frac{b_n}{n}\Big)^{\frac{md(p-1)}{2p}} \mathbb{E}\Big[\int_{\mathbb{R}^d} f\Big(\sqrt{\frac{b_n}{n}} x\Big) l(n, [x]) dx\Big]^m. \end{aligned}$$

Therefore,

$$\begin{aligned} &\sum_{m=0}^\infty \frac{1}{m!} \theta^m \Big(\frac{b_n}{n}\Big)^{\frac{2p-d(p-1)}{2p} m} \big(\mathbb{E}\, I_n^m\big)^{1/p} \\ &\geq \mathbb{E} \exp\Big\{\theta \frac{b_n}{n} \int_{\mathbb{R}^d} f\Big(\sqrt{\frac{b_n}{n}} x\Big) l(n, [x]) dx\Big\} \\ &\sim \mathbb{E} \exp\Big\{\theta \frac{b_n}{n} \sum_{k=0}^n f\Big(\sqrt{\frac{b_n}{n}} S(k)\Big)\Big\}. \end{aligned}$$

To deal with the right hand side, we need a discrete version (Theorem 4.1, Chen and Li (2004)) of the Feynman-Kac large deviation. Similar to (2.6),

$$\begin{aligned} &\liminf_{n\to\infty} \frac{1}{b_n} \log \mathbb{E} \exp\Big\{\theta \frac{b_n}{n} \sum_{k=0}^n f\Big(\sqrt{\frac{b_n}{n}} S(k)\Big)\Big\} \\ &\geq \sup_{g \in \mathcal{F}_d} \Big\{\theta \int_{\mathbb{R}^d} f(x) g^2(x) dx - \frac{1}{2} \int_{\mathbb{R}^d} \langle \nabla g(x), \Gamma \nabla g(x) \rangle dx\Big\}. \end{aligned}$$

Consequently,

$$\begin{aligned} &\liminf_{n\to\infty} \frac{1}{b_n} \log \sum_{m=0}^\infty \frac{\theta^m}{m!} \Big(\frac{b_n}{n}\Big)^{\frac{2p-d(p-1)}{2p} m} \Big(\mathbb{E}\, I_n^m\Big)^{1/p} \\ &\geq \sup_{g \in \mathcal{F}_d} \Big\{\theta \int_{\mathbb{R}^d} f(x) g^2(x) dx - \frac{1}{2} \int_{\mathbb{R}^d} \langle \nabla g(x), \Gamma \nabla g(x) \rangle dx\Big\}. \end{aligned}$$

Taking supremum over $||f||_q = 1$ on the right hand side gives

$$\liminf_{n\to\infty} \frac{1}{b_n} \log \sum_{m=0}^{\infty} \frac{\theta^m}{m!} \Big(\frac{b_n}{n}\Big)^{\frac{2p-d(p-1)}{2p} m} \Big(\mathbb{E}\, I_n^m\Big)^{1/p}$$
$$\geq \sup_{g\in\mathcal{F}_d} \bigg\{\theta\bigg(\int_{\mathbb{R}^d} |g(x)|^{2p} dx\bigg)^{1/p} - \frac{1}{2}\int_{\mathbb{R}^d} \langle \nabla g(x), \Gamma \nabla g(x)\rangle dx\bigg\}.$$

Finally, the lower bound of (5.33) follows from the analytic fact (Lemma A.2, Chen (2004)) that the right hand side of the above inequality is equal to the right hand side of (5.33).□

To compare with the law of the iterated logarithm (LIL) given in the critical dimensions, we apply Theorem 5.4 and 5.5 to the LIL by taking $b_n = \lambda \log\log n$. The technical involvement is a essentially standard practice of Borel-Cantelli lemma.

Theorem 5.6. *Assume $p(d-2) < d$. For the intersection local time I_n,*

$$\limsup_{n\to\infty} n^{-\frac{2p-d(p-1)}{2}} (\log\log n)^{-\frac{d(p-1)}{2}} I_n$$
$$= \Big(\frac{2}{p}\Big)^{-\frac{d(p-1)}{2}} \det(\Gamma)^{-\frac{p-1}{2}} \kappa(d,p)^{2p} \quad a.s. \tag{5.34}$$

For the range intersection J_n, we have

(1). As $d = 1$ and $p \geq 2$

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n \log\log n}} J_n = \sqrt{\frac{2}{p}}\sigma \quad a.s. \tag{5.35}$$

(2). As $d = 2$ and $p \geq 2$,

$$\limsup_{n\to\infty} \frac{(\log n)^p}{n(\log\log n)^{p-1}} J_n = (2\pi)^p \Big(\frac{2}{p}\Big)^{p-1} \sqrt{\det(\Gamma)}\kappa(2,p)^{2p} \quad a.s. \tag{5.36}$$

(3). As $d = 3$ and $p = 2$,

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n(\log\log n)^3}} J_n = \gamma_S^2 \det(\Gamma)^{-1/2} \kappa(3,2)^4 \quad a.s. \tag{5.37}$$

We now come to the super-critical dimensions defined by $p(d-2) > d$ and discuss the tail behaviors of I_∞ and J_∞ defined by (5.1) and (5.2), respectively. The investigation of I_∞ and J_∞ in the super-critical dimensions goes back to an influential paper by Khanin, Mazel, Shlosman and Sinai (1994) where it is shown that in the special case $p=2$, $d\geq 5$,

$$\exp\{-c_1\lambda^{1/2}\}\leq \mathbb{P}\{I_\infty\geq\lambda\}\leq\exp\{-c_2\lambda^{1/2}\} \tag{5.38}$$

$$\exp\Big\{-\lambda^{\frac{d-2}{d}+\delta}\Big\}\leq \mathbb{P}\{J_\infty\geq\lambda\}\leq\exp\Big\{-\lambda^{\frac{d-2}{d}-\delta}\Big\} \tag{5.39}$$

for large $\lambda>0$.

The discovery of Khanin *et al* sharply contrasts the results stated in Theorem 5.1, and the "$d=3$" part in Theorem 5.4 and 5.5, which suggest that the intersection of independent ranges behaves like a constant multiple of the intersection local time in the high dimension ($d\geq 3$). The challenge lies in understanding the difference of these behaviors, providing sharp estimates for the tails, and understanding the underlying "optimal strategies".

The exact tail of I_∞ was recently obtained in Chen and Mörters (2007). For the purpose of comparison we also consider the setting of the random walks with lattice values but continuous time. Let $X_1(t),\cdots,X_p(t)$ be independent symmetric random walks on $\mathbb{Z}^d$ with same distribution. Under $p(d-2)>d$ we write

$$\mathcal{I}_\infty=\int_0^\infty\cdots\int_0^\infty 1_{\{X_1(t_1)=\cdots=X_p(t_p)\}}dt_1\cdots dt_p.$$

Theorem 5.7. *Under $p(d-2)>d$,*

$$\lim_{\lambda\to\infty}\frac{1}{\lambda^{1/p}}\log\mathbb{P}\{\mathcal{I}_\infty\geq\lambda\}=-\frac{p}{\rho} \tag{5.40}$$

while

$$\lim_{\lambda\to\infty}\frac{1}{\lambda^{1/p}}\log\mathbb{P}\{I_\infty\geq\lambda\}=-p\gamma^{-1}(1) \tag{5.41}$$

where

$$\rho=\sup\Big\{\sum_{x,y\in\mathbb{Z}^d}G(x-y)f(x)f(y);\quad \sum_{x\in\mathbb{Z}^d}|f(x)|^{\frac{2p}{2p-1}}=1\Big\} \tag{5.42}$$

$$G(x)=\int_0^\infty\mathbb{P}\{X_1(t)=x\}dt\qquad x\in\mathbb{Z}^d; \tag{5.43}$$

and where $\gamma^{-1}(b)$ is the inverse of the function

$$\gamma(\theta)=\sup\Big\{\sum_{x,y\in\mathbb{Z}^d}G_1(x-y)\sqrt{\Big(e^{\theta f(x)}-1\Big)\Big(e^{\theta f(y)}-1\Big)}g(x)g(y);\\ f,g\geq 0 \text{ and } \sum_{x\in\mathbb{Z}^d}f^{\frac{p}{p-1}}(x)=\sum_{x\in\mathbb{Z}^d}g^2(x)=1\Big\}\qquad\theta>0 \tag{5.44}$$

$$G_1(x) = \sum_{k=1}^{\infty} \mathbb{P}\{S_1(k) = x\} \qquad x \in \mathbb{Z}^d. \tag{5.45}$$

It is interesting to see some unexpected difference in variation between the case of continuous time and the case of discrete time. The approach for Theorem 5.7 is the method of high moment asymptotics, which will be discussed in section 7.

The problem on the exact tail for J_∞ is harder and still remains open. Perhaps the most important progress since the paper Khanin *et al* was made by van den Berg, Bolthausen and den Hollander (2004): To apply the famous Donsker-Varadhan large deviations, they consider the intersection of two independent Brownian sausages $W_1^\epsilon(t)$ and $W_2^\epsilon(t)$

$$W_i^\epsilon(t) = \bigcup_{0\le s\le t} \{x \in \mathbb{R}^d;\ |x - W_i(s)| \le \epsilon\} \qquad i = 1, 2$$

instead of $S_1(0,\infty)$ and $S_2(0,\infty)$ and show that as $d \ge 5$, the volume $|W_1^\epsilon(\theta t) \cap W_2^\epsilon(\theta t)|$ of the intersection of $W_1^\epsilon(t)$ and $W_2^\epsilon(t)$ has the the following tail behavior: as

$$\lim_{t\to\infty} t^{-\frac{d-2}{d}} \log \mathbb{P}\Big\{|W_1^\epsilon(\theta t) \cap W_2^\epsilon(\theta t)| \ge t\Big\} = -I_d^\epsilon(\theta) \tag{5.46}$$

where the rate function $I_d^\epsilon(\cdot)$ is given in terms of the variation

$$I_d^\epsilon(\theta) = \theta \inf_{\phi\in\Phi_d(\theta)} \int_{\mathbb{R}^d} |\nabla\phi(x)|^2 dx \tag{5.47}$$

and

$$\begin{aligned}\Phi_d(\theta) =&\Big\{\phi \in W^{1,2}(\mathbb{R}^d);\quad \int_{\mathbb{R}^2} \phi^2(x)dx = 1,\ \text{and}\\ &\int_{\mathbb{R}^d} \Big(1 - \exp\Big\{-2\epsilon^{d-2}\pi^{d/2}\theta\Gamma\Big(\frac{d-2}{2}\Big)^{-1}\phi^2(x)\Big\}\Big)^2 dx \ge 1\Big\}.\end{aligned} \tag{5.48}$$

They also show that there exists a critical $\theta^* > 0$ such that

$$I_d^\epsilon(\theta) = \frac{\Gamma\Big(\dfrac{d-2}{d}\Big)}{2\epsilon^{d-2}\pi^{d/2}} \inf\Big\{||\nabla\psi||_2^2;\quad ||\psi||_2 = 1,\ \text{and}\ \int_{\mathbb{R}^d} \big(1 - e^{-\psi^2(x)}\big)^2 dx = 1\Big\}$$

for all $\theta \ge \theta^*$. This strongly suggests (conjectured in van den Berg *et al* (2004)) that

$$\lim_{t\to\infty} t^{-\frac{d-2}{d}} \log \mathbb{P}\Big\{|W_1^\epsilon(\infty) \cap W_2^\epsilon(\infty)| \ge t\Big\} = -I_d^\epsilon(\theta^*). \tag{5.49}$$

Turning back to the setting of random walks, it becomes natural to conjecture that for the intersection J_∞ of p independent ranges given in (5.2), the limit

$$\lim_{t\to\infty} t^{-\frac{d-2}{d}} \log \mathbb{P}\big\{J_\infty \ge t\big\} = -C(d,p) \tag{5.50}$$

is non-trivial, and to ask for the identification of the constant $C(d,p)$.

6. Self-intersection local time and range of single random walk.

Through this section, $\{S(n)\}$ is a symmetric, square integrable random walk on $\mathbb{Z}^d$. We assume that the smallest subgroup of $\mathbb{Z}^d$ that supports $\{S(n)\}$ is $\mathbb{Z}^d$ itself. We use the notation Γ for the covariance matrix of $\{S(n)\}$, and σ^2 for the variance of $\{S(n)\}$ in the case $d=1$. We set $\gamma_S = \mathbb{P}\{S(n)\neq 0 \ \forall n\geq 1\}$ when $d\geq 3$.

The self-intersection is closely related to the range. Roughly speaking, the more intense the self-intersection is, the less spread out the sample path becomes. As $d=1$, the p-multiple self-intersection local time is essentially same as

$$\frac{1}{p!}\sum_{x\in\mathbb{Z}} l^p(n,x)$$

where $l(n,x)$ is the local time of the random walk $\{S(n)\}$.

The following result appears as a discrete version Theorem 4.1 and was obtained by Chen and Li (2004).

Theorem 6.1. *Let $d=1$. For any $p\geq 2$ and for any positive sequence satisfying*

$$b_n \longrightarrow \infty \text{ and } b_n = o(n) \quad (n\to\infty). \tag{6.1}$$

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\Big\{\sum_{x\in\mathbb{Z}} l^p(n,x)\geq n^{\frac{p+1}{2}} b_n^{\frac{p-1}{2}}\Big\}\\ &= -\sigma^2\frac{1}{4(p-1)}\Big(\frac{p+1}{2}\Big)^{\frac{3-p}{p-1}} B\Big(\frac{1}{p-1},\frac{1}{2}\Big).\end{aligned} \tag{6.2}$$

As for the range, we have (Chen (2006))

Theorem 6.2. *Let $d=1$. For any positive sequence satisfying (6.1),*

$$\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\Big\{\#\{S[0,n]\}\geq\sqrt{nb_n}\Big\} = -\frac{1}{2\sigma^2} \tag{6.3}$$

$$\lim_{n\to\infty}\frac{1}{b_n}\log\mathbb{P}\Big\{\#\{S[0,n]\}\leq\sqrt{\frac{n}{b_n}}\Big\} = -\frac{\pi^2\sigma^2}{2}. \tag{6.4}$$

To see how the self-intersection connect to the range in the case $d=1$, we perform the following rough but simple comparison:

$$n=\sum_{x\in\mathbb{Z}} l(n,x) = \sum_{x\in S[0,n]} l(n,x)\leq\Big(\#\{S[0,n]\}\Big)^{1/q}\Big(\sum_{x\in\mathbb{Z}} l^p(n,x)\Big)^{1/p}.$$

In view of (6.2) and (6.4), the above comparison captures the relation between self-intersection and range with accuracy up to constant.

In the case $d \geq 2$, concentration becomes a dominating force. Dvoretzky and Erdös proved a strong law of large numbers

$$\#\big\{S[0,n]\big\}/\mathbb{E}\,\#\big\{S[0,n]\big\} \longrightarrow 1 \qquad a.s. \tag{6.5}$$

Due to the difficulty brought by the dimension multiplicity, the current study of self-intersection local time in the case $d \geq 2$ focuses on the case $p = 2$. From now on

$$Q_n = \sum_{1\leq j<k\leq n} 1_{\{S(j)=S(k)\}}. \tag{6.6}$$

The law of the large numbers also holds for Q_n. The reason behind is that when the dimension gets higher, the short-range intersection gets bigger share in total intersection. When $d \geq 2$, one can pick a suitable $l = l(n)$ such that $l = o(n)$, $l(n) \to \infty$ as $n \to \infty$ and that in both decompositions (by the classic inclusion-exclusion formula)

$$\#\big\{S(0,n]\big\} = \sum_{i=1}^{l} \#\Big\{S\Big(\frac{i-1}{l}n, \frac{i}{l}n\Big]\Big\} - \sum_{i=2}^{l} \#\Big\{S\Big(0, \frac{i-1}{l}n\Big] \cap S\Big(\frac{i-1}{l}n, \frac{i}{l}n\Big]\Big\} \tag{6.7}$$

and

$$Q_n = \sum_{i=1}^{l} \sum_{\frac{i-1}{l}n<j,k\leq \frac{i}{l}n} 1_{\{S(j)=S(k)\}} + \sum_{i=2}^{l} \sum_{j=1}^{\frac{i-1}{l}n} \sum_{k=\frac{i-1}{l}n+1}^{\frac{i}{l}n} 1_{\{S(j)=S(k)\}} \tag{6.8}$$

the first summation (short-range intersection) dominates. Notice that the terms in the first summation are independent. So the classic law of large numbers essentially applies here.

Due to concentration phenomenon, the renormalized quantities

$$\#\big\{S(0,n]\big\} - \mathbb{E}\,\#\big\{S(0,n]\big\} \quad \text{and} \quad Q_n - \mathbb{E}\,Q_n$$

naturally become the objects of the investigation. We now list the expected values of $\#\big\{S(0,n]\big\}$ and Q_n below for comparison

$$\mathbb{E}\,Q_n \sim \begin{cases} \dfrac{1}{2\pi\sqrt{\det(\Gamma)}} n\log n & d=2 \\ (\gamma_S^{-1}-1)n & d\geq 3; \end{cases} \tag{6.9}$$

$$\mathbb{E}\,\#\big\{S(0,n]\big\} \sim \begin{cases} 2\pi\sqrt{\det(\Gamma)}\dfrac{n}{\log n} & d=2 \\ \gamma_S n & d\geq 3. \end{cases} \tag{6.10}$$

Renormalization reduces the contribution from short-range intersection. In the case $d \geq 3$, however, the long range intersection is so weak that the short-range intersection still dominates even after being renormalized. Consequently, the classic central limit theorem implies that the sequences

$$\frac{\#\big\{S(0,n]\big\} - \mathbb{E}\,\#\big\{S(0,n]\big\}}{\sqrt{\mathrm{Var}\,\big(\#\big\{S(0,n]\big\}\big)}} \quad \text{and} \quad \frac{Q_n - \mathbb{E}\,Q_n}{\sqrt{\mathrm{Var}\,(Q_n)}}$$

are asymptotically standard normal. The reader is referred to the papers Orey and Jain for the case $d \geq 5$, and Jain and Pruitt (1971) for the case $d = 3, 4$.

Historically, the study in the case $d \geq 3$ has been focused on the renormalized range $\#\big\{S(0,n]\big\} - \mathbb{E}\,\#\big\{S(0,n]\big\}$ due to the fact that the quantities

$$\mathrm{Var}\,\big(\#\big\{S(0,n]\big\}\big) \quad \text{and} \quad \mathrm{Var}\,(Q_n)$$

differ asymptotically only by a constant multiplier (it seems that this fact has been known by the people in the area for a long time without being explicitly stated, see Lemma 5.1, Chen (2007c) for a proof). Consequently, the renormalized range and renormalized self-intersection local time differ asymptotically by a constant multiplier when $d \geq 3$.

The computation of the variance of the range can be found in Jain and Orey (1968), Jain and Pruitt (1971). $\mathrm{Var}\,\big(\#\big\{S(0,n]\big\}\big)$ is of the order $n \log n$ when $d = 3$ and of the order n when $n \geq 4$. Not surprisingly, $\#\big\{S(0,n]\big\} - \mathbb{E}\,\#\big\{S(0,n]\big\}$ yields a Gaussian tail. This fact was proved in Jain-Pruitt (1972) and Bass-Kumagai (2002) in the form of law of the iterated logarithm or almost sure invariance principle.

Theorem 6.3. *(Jain and Pruitt) When $d \geq 4$,*

$$\limsup_{n\to\infty} \frac{\#\big\{S(0,n]\big\} - \mathbb{E}\,\#\big\{S(0,n]\big\}}{\sqrt{2n \log\log n}} = \gamma_S \lambda_0 \qquad a.s. \tag{6.11}$$

$$\liminf_{n\to\infty} \frac{\#\big\{S(0,n]\big\} - \mathbb{E}\,\#\big\{S(0,n]\big\}}{\sqrt{2n \log\log n}} = -\gamma_S \lambda_0 \qquad a.s. \tag{6.12}$$

where

$$\lambda_0 = \sqrt{G^2(0) + G(0) + 2\sum_{x\in\mathbb{Z}^d} G^3(x)}$$

and

$$G(x) = \sum_{k=1}^{p} \mathbb{P}\{S(k) = x\}$$

Theorem 6.4. *(Bass and Kumagai) When $d = 3$,*

$$\limsup_{n\to\infty} \frac{\#\{S(0,n]\} - \mathbb{E}\,\#\{S(0,n]\}}{\sqrt{n\log n\log\log n}} = \frac{\gamma_S}{\pi}\sqrt{\det\Gamma} \qquad a.s. \tag{6.13}$$

$$\liminf_{n\to\infty} \frac{\#\{S(0,n]\} - \mathbb{E}\,\#\{S(0,n]\}}{\sqrt{n\log n\log\log n}} = -\frac{\gamma_S}{\pi}\sqrt{\det\Gamma} \qquad a.s. \tag{6.14}$$

In the same paper, Bass and Kumagai also obtained a partial result in the case $d = 2$. They prove under additional condition that

$$\limsup_{n\to\infty} \frac{(\log n)^2}{n\log\log\log n}\Big(\#\{S(0,n]\} - \mathbb{E}\,\#\{S(0,n]\}\Big) = C \qquad a.s. \tag{6.15}$$

with the unidentified constant C.

The case $d = 2$ is the most challenging and interesting case. First, renormalization changes the whole dynamics of the system. The long-range intersection represented by the second summations in (6.7) and (6.8) becomes the dominating force after renormalization. Consequently, the renormalized self-intersection local time and renormalized range have quite different limiting behaviors. In addition to the difference in magnitude caused by recurrence, there is a sign switch due to the difference in sign of the second terms in the decompositions (6.7) and (6.8). Notice also that the terms in the second summations of both (6.7) and (6.8) are strongly correlated so the long-range intersection parts yield a non-Gaussian limit. Remarkably, Le Gall (1986a) observed that the long-range intersection part in both (6.7) and (6.8) are essentially attracted by the renormalized self-intersection local time $\gamma\big([0,1]^2_<\big)$ of a planar Brownian motion (see section 4). Consequently, Le Gall proved that

$$\frac{Q_n - \mathbb{E}\,Q_n}{n} \xrightarrow{d} \det(\Gamma)^{-1/2}\gamma\big([0,1]^2_<\big) \tag{6.16}$$

$$\frac{(\log n)^2}{n}\Big(\#\{S(0,n]\} - \mathbb{E}\,\#\{S(0,n]\}\Big) \xrightarrow{d} -4\pi^2\sqrt{\det(\Gamma)}\gamma\big([0,1]^2_<\big). \tag{6.17}$$

The tail probabilities for the renormalized self-intersection local time and for the renormalized range were computed in Bass, Chen and Rosen (2006, 2007). To compare them with Theorem 6.3 and Theorem 6.4, we state them in terms of the law of the iterated logarithm.

Theorem 6.5. *When $d = 2$,*

$$\limsup_{n\to\infty} \frac{Q_n - \mathbb{E}\,Q_n}{n\log\log n} = \frac{\kappa(2,2)^4}{\sqrt{\det(\Gamma)}} \qquad a.s. \tag{6.18}$$

$$\liminf_{n\to\infty} \frac{Q_n - \mathbb{E}\,Q_n}{n\log\log\log n} = \frac{1}{2\pi\sqrt{\det(\Gamma)}} \qquad a.s. \tag{6.19}$$

$$\limsup_{n\to\infty} \frac{(\log n)^2}{n\log\log\log n}\Big(\#\{S(0,n]\} - \mathbb{E}\,\#\{S(0,n]\}\Big) = 2\pi\sqrt{\det(\Gamma)} \qquad a.s. \tag{6.20}$$

$$\liminf_{n\to\infty} \frac{(\log n)^2}{n\log\log n}\Big(\#\{S(0,n]\} - \mathbb{E}\,\#\{S(0,n]\}\Big) = -(2\pi)^2\sqrt{\det(\Gamma)}\kappa(2,2)^4 \qquad a.s. \tag{6.21}$$

where $\kappa(2,2)$ is the best constant of Gagliardo-Nirenberg inequality given in (3.4) (with $d = p = 2$).

The connection between Theorem 4.4 and Theorem 6.5 is clearly visible. In particular, it is not hard to understand where the non-symmetric behavior appearing in Theorem 6.5 comes from. It is also interesting to see that the limsup for $Q_n - \mathbb{E}\,Q_n$ is relevant to the liminf for $\#\{S[0,n]\} - \mathbb{E}\,\#\{S[0,n]\}$, while the limsup for $\#\{S[0,n]\} - \mathbb{E}\,\#\{S[0,n]\}$ is relevant to the liminf for $Q_n - \mathbb{E}\,Q_n$.

Based on the integrability of I_n and J_n established in Theorem 5.3 (with $d = p = 2$), a discrete version of Theorem 4.3 was obtained in Bass, Chen and Rosen (2006). The approach appears to be a modification of the one used for Theorem 4.3.

Theorem 6.6. *Let $d = 2$. There is $\theta > 0$ such that*

$$\sup_{n\ge1} \mathbb{E}\exp\Big\{\theta\frac{1}{n}\big(Q_n - \mathbb{E}\,Q_n\big)\Big\} < \infty \tag{6.22}$$

$$\sup_{n\ge1} \mathbb{E}\exp\Big\{-\theta\frac{(\log n)^2}{n}\Big(\#\{S(0,n]\} - \mathbb{E}\,\#\{S(0,n]\}\Big)\Big\} < \infty. \tag{6.23}$$

On the other hand, by the fact (Theorem 4.4) that

$$\mathbb{E}\exp\Big\{\theta\gamma\big([0,1]^2_<\big)\Big\} = \infty \qquad \theta > \kappa(2,2)^{-4}$$

and by the weak laws given in (6.16) and (6.17), we have

$$\sup_{n\ge1} \mathbb{E}\exp\Big\{\theta\frac{1}{n}\big(Q_n - \mathbb{E}\,Q_n\big)\Big\} = \infty$$

for $\theta > \sqrt{\det(\Gamma)}\kappa(2,2)^{-4}$; and

$$\sup_{n\ge1} \mathbb{E}\exp\Big\{-\theta\frac{(\log n)^2}{n}\Big(\#\{S(0,n]\} - \mathbb{E}\,\#\{S(0,n]\}\Big)\Big\} = \infty$$

for every $\theta > (4\pi^2)^{-1}\det(\Gamma)^{-1/2}\kappa(2,2)^{-4}$.

A natural question is to find the critical exponents. We may ask, for example, if $\theta = \sqrt{\det(\Gamma)}\kappa(2,2)^{-4}$ is the critical exponent for $Q_n - \mathbb{E}\,Q_n$. The following example (originally given in Bass, Chen and Rosen (2006)) gives a negative answer.

Example 6.1. Let N be an arbitrarily large integer and write $\epsilon = 2/N^2$. Let the random walk $\{S(n)\}$ is the partial sum of the i.i.d sequence of random vectors in $\mathbb{Z}^d$ that take the values $(N,0), (-N,0), (0,N), (0,-N)$ with probability $\epsilon/4$, and take the value $(0,0)$ with probability $1-\epsilon$. Then Γ is identity matrix. The event that $S(k)=0$ for all $k \le n$ has probability at least $(1-\epsilon)^n$. On this event $Q_n = n(n-1)/2$. Hence,

$$\mathbb{E}\exp\Big\{\theta\frac{1}{n}\big(Q_n - \mathbb{E}\,Q_n\big)\Big\} \ge \exp\Big\{-\theta\frac{\mathbb{E}\,Q_n}{n}\Big\}(1-\epsilon)^n\exp\Big\{\theta\frac{n-1}{2}\Big\}.$$

So the critical exponent for $Q_n - \mathbb{E}\,Q_n$ is no more than $2\log(1-\epsilon)^{-1}$. Since ϵ can be arbitrarily small, so is the critical exponent.

This example shows the critical exponent in the continuous setting does not pass to the discrete setting. The identification of these critical exponents in the case of random walks remains open.

Some recent papers have shown interest in the tail behaviors of the self-intersection local times which are not governed by the weak law of convergence. As is well known that $\mathbb{E}\,Q_n$ is of order n as $d \ge 3$. Therefore, the investigation of the tail probability

$$\mathbb{P}\{Q_n \ge yn\}$$

is related to the law of large numbers given in (6.5). Some recent investigation suggests an interesting dimension dependence pattern. Asselah (2006) recently proved that as $d=3$

$$\exp\big\{-c_1 y^{2/3} n^{1/3}\big\} \le \mathbb{P}\Big\{\sum_{x\in\mathbb{Z}^d} l^2(n,x) \ge yn\Big\} \le \exp\big\{-c_2 y^{-2/3} n^{1/3}\big\}$$

for y large enough. As $d \ge 5$, it is shown in Asselah and Castell (2006) that for y large enough there are $c_1, c_2 > 0$ (depending on y) such that

$$\exp\big\{-c_1\sqrt{n}\big\} \le \mathbb{P}\Big\{\sum_{x\in\mathbb{Z}^d} l^2(n,x) \ge yn\Big\} \le \exp\big\{-c_2\sqrt{n}\big\}.$$

The case $d=4$ remains open. Further, one may ask for a sharper estimate of the above tails.

We now mention a recent work (Chen (2007c)) on a model called charged polymer. A lattice polymer is often described by physicists as interpolation line segment with the vertexes given as the n-step lattice (simple) random walk

$$\{S(1), \cdots, S(n)\}.$$

By placing independent, identically distributed electric charges $\omega_k = \pm 1$ to each vertex of the polymer, Kantor and Kardar (1991) consider a model of polymers with random electrical charges associated with the Hamiltonian

$$H_n = \sum_{1\le j<k\le n} \omega_j\omega_k 1_{\{S(j)=S(k)\}}. \tag{6.24}$$

If we assume that when two charges meet, the pair with opposite sign gives negative contribution while the pair with same sign gives positive contribution, then H_n represents the total electrical interaction charge of the polymer $\{S(1), \cdots, S(n)\}$.

We point out some other works by physicists in this direction. In Derrida, Griffiths and Higgs (1992), the charges are i.i.d. Gaussian variables. In Derrida and Higgs (1994), the charges take $0-1$ values. We also refer the reader to Martinez and Petritis (1996); Buffet and Pulé (1997) for the continuous versions of the polymer with random charges. In particular, we cite the comment by Martinez and Petritis (1996): "It is argued that a protein molecule is very much like a random walk with random charges attached at the vertexes of the walk; these charges are interacting through local interactions mimicking Lennard-Jones or hydrogen-bond potentials".

The central limit theorems and moderate deviations for H_n have been established in the recent work Chen (2007c). The central limit theorems state:

When $d=1$,

$$\frac{1}{n^{3/4}}H_n \xrightarrow{d} (2\sigma)^{-1/2}\Big(\int_{-\infty}^{\infty} L^2(1,x)dx\Big)^{1/2} U \tag{6.25}$$

where U is a random variable with standard normal distribution, $L(t,x)$ is the local time of the 1-dimensional Brownian motion $W(t)$ such that U and $W(t)$ are independent.

When $d=2$,

$$\frac{1}{\sqrt{n\log n}}H_n \xrightarrow{d} \frac{1}{\sqrt{2\pi}\sqrt[4]{\det\Gamma}}U. \tag{6.26}$$

When $d\ge 3$,

$$\frac{1}{\sqrt{n}}H_n \xrightarrow{d} \sqrt{\gamma_S}U. \tag{6.28}$$

Not surprisingly, it is the limiting random variables in the laws of of weak convergence who decide the tail behaviors of H_n. Therefore we have the following theorem on the moderate deviations for H_n (Chen (2007c)).

Theorem 6.7. *As $d = 1$,*

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{ \pm H_n \ge \lambda (nb_n)^{3/4} \Big\} = -\frac{1}{2}\sigma^{2/3}(3\lambda)^{4/3} \qquad (\lambda > 0) \tag{6.29}$$

for any positive sequence $\{b_n\}$ satisfying

$$b_n \longrightarrow \infty \quad \text{and} \quad b_n = o(\sqrt[7]{n}) \qquad (n \to \infty).$$

As $d = 2$,

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{ \pm H_n \ge \lambda \sqrt{n(\log n) b_n} \Big\} = -\pi\sqrt{\det(\Gamma)}\lambda^2 \qquad (\lambda > 0) \tag{6.30}$$

for any positive sequence $\{b_n\}$ satisfying

$$b_n \longrightarrow \infty \quad \text{and} \quad b_n = o(\log n) \qquad (n \to \infty).$$

As $d \ge 3$,

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}\Big\{ \pm H_n \ge \lambda \sqrt{nb_n} \Big\} = -\frac{\lambda^2}{2\gamma} \qquad (\lambda > 0) \tag{6.31}$$

for any positive sequence $\{b_n\}$ satisfying

$$b_n \longrightarrow \infty \quad \text{and} \quad b_n = o\Big(\frac{n^{1/3}}{(\log n)^{4/3}}\Big) \qquad (n \to \infty).$$

7. Method of high moment asymptotics.

The conventional approach dealing with the tail probabilities of intersection local times is Feynman-Kac formula. This method is also closely related to the results and tools developed along the line of Donsker-Varadhan large deviation theory. In this section we provide a new alternative known as the *method of high moment asymptotics*. Moment method has been extensively used in the study of limit laws of intersection local times and related models, due to the fact that the integer moments of intersection local times often have reasonably nice representations — see (3.15) for example. In Le Gall (1986a, 1986b), Rosen (1990), Le Gall and Rosen (1991), the law of weak convergence is obtained by establishing the convergence between the moments. In this case the power of the moment is arbitrary but fixed. According to Lemma 3.1, on the other hand, the power of the moment goes to infinity when it comes to large deviations. We call this type of the problems high moment asymptotics.

Here we take the proof of (5.40) in Theorem 5.7 as example. The conventional technique does not allow the treatment of infinite time horizon. So we compute the high moment of $\mathcal{I}_\infty$. Write

$$\mathcal{I}_\infty = \sum_{x\in\mathbb{Z}^d}\prod_{j=1}^p\int_0^\infty 1_{\{X_j(t)=x\}}dt.$$

We have that for any $m \geq 1$,

$$\begin{aligned}
\mathbb{E}\,\mathcal{I}_\infty^m &= \sum_{x_1,\cdots,x_m\in\mathbb{Z}^d}\Big[\int_0^\infty\cdots\int_0^\infty \mathbb{P}\{X(t_1)=x_1,\cdots,X(t_m)=x_m\}dt_1\cdots dt_m\Big]^p\\
&= \sum_{x_1,\cdots,x_m\in\mathbb{Z}^d}\Big[\sum_{\sigma\in\Sigma_m}\int_{\{t_1\leq\cdots\leq t_m\}}\mathbb{P}\{x(t_1)=x_{\sigma(1)},\cdots,X(t_m)=x_{\sigma(m)}\}dt_1\cdots dt_m\Big]^p\\
&= \sum_{x_1,\cdots,x_m\in\mathbb{Z}^d}\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m G(x_{\sigma(k)}-x_{\sigma(k-1)})\Big]^p
\end{aligned} \tag{7.1}$$

where Σ_m is the permutation group on $\{1,\cdots,m\}$. By Lemma 3.1 we need to establish

$$\lim_{m\to\infty}\frac{1}{m}\log\sum_{x_1,\cdots,x_m\in\mathbb{Z}^d}\Big[\frac{1}{m!}\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m G(x_{\sigma(k)}-x_{\sigma(k-1)})\Big]^p = p\log\rho. \tag{7.2}$$

The method of high moment asymptotics first appeared in König-Mörters (2002). The following Theorem 7.1 and 7.2 are modified versions given in Chen and Mörters (2007).

Theorem 7.1. *Let $p \geq 1$ be a constant, let $(\Omega,\mathcal{A},\pi)$ be a measure space and let K: $\Omega\times\Omega \longrightarrow \mathbb{R}^+$ be a measurable, non-negative function satisfying the following assumptions:*

(1). Symmetry: $K(x,y)=K(y,x)$ for any $x,y\in\Omega$.

(2). Irreducibility: For any $x\in\Omega$,

$$\pi\Big(\{y\in\Omega;\quad G(x,y)=0\}\Big)=0$$

(3). Double integrability: For any $g\in\mathcal{L}^{\frac{2p}{2p-1}}(\Omega,\mathcal{A},\pi)$,

$$\iint_{\Omega\times\Omega}K(x,y)g(x)g(y)\pi(dx)\pi(dy)<\infty.$$

Then

$$\begin{aligned}
&\liminf_{m\to\infty}\frac{1}{m}\log\int_{\Omega^m}\pi(dx_1)\cdots\pi(dx_m)\Big[\frac{1}{m!}\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)},x_{\sigma(k)})\Big]^p\\
&\geq p\log\sup_g\iint_{\Omega\times\Omega}K(x,y)g(x)g(y)\pi(dx)\pi(dy)
\end{aligned} \tag{7.3}$$

where the supremum is taken over all functions g on Ω satisfying

$$\int_\Omega |g(x)|^{\frac{2p}{2p-1}}\pi(dx) = 1.$$

The key in the proof of Theorem 7.1 is to remove permutation. By Hölder inequality:

$$\begin{aligned}
&\left\{\int_{\Omega^m}\pi(dx_1)\cdots\pi(dx_m)\left[\frac{1}{m!}\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)},x_{\sigma(k)})\right]^p\right\}^{1/p}\\
&\ge \int_{\Omega^m}\pi(dx_1)\cdots\pi(dx_m)\Big(\prod_{k=1}^m f(x_k)\Big)\frac{1}{m!}\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)},x_{\sigma(k)})\\
&= \int_{\Omega^m}\pi(dx_1)\cdots\pi(dx_m)\prod_{k=1}^m K(x_{k-1},x_k)f(x_k)
\end{aligned}$$

where the test function $f \ge 0$ satisfies $||f||_q = 1$ with $q > 1$ conjugate to p. In this way, the proof is reduced to a typical principal eigenvalue problem followed by optimization of f.

The upper bound is much harder. Indeed, we are able to establish it only when the state space Ω is finite.

Theorem 7.2. *Let Ω be a finite set and π: $\Omega \longrightarrow \mathbb{R}^+$ and K: $\Omega\times\Omega \longrightarrow \mathbb{R}^+$. Assume that $K(x,y) = K(y,x)$. Then*

$$\limsup_{m\to\infty}\frac{1}{m}\log\sum_{x_1,\cdots,x_n\in\Omega}\Big(\prod_{k=1}^m \pi(x_k)\Big)\Big[\frac{1}{m!}\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m K(x_{\sigma(k-1)},x_{\sigma(k)})\Big]^p \le p\log\hat\rho \tag{7.4}$$

where Σ_m is the permutation group on $\{1,\cdots,m\}$ and

$$\hat\rho = \sup\Big\{\sum_{x,y\in\Omega}K(x,y)g(x)g(y)\pi(x)\pi(y);\quad \sum_{x\in\Omega}|g(x)|^{\frac{2p}{2p-1}}\pi(x) = 1\Big\}.$$

The lower bound given in Theorem 7.1 has enough generality for practical use. The proof of Theorem 7.2 is combinatorial and the assumption on the finite state space is essential to the argument currently used for Theorem 7.2. The challenge we face is to extend Theorem 7.2 to the setting of reasonable generality.

Going back to the proof of Theorem 5.8, we need only to establish the upper bound of (7.2). By performing a projection on a torus, we have that for any integer $N \ge 1$,

$$\begin{aligned}
&\sum_{x_1,\cdots,x_m\in\mathbb{Z}^d}\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m G(x_{\sigma(k)}-x_{\sigma(k-1)})\Big]^p\\
&\le \sum_{y_1,\cdots,y_m\in[-N,N)^d}\Big[\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m \widetilde{G}_N(y_{\sigma(k)}-y_{\sigma(k-1)})\Big]^p
\end{aligned}\tag{7.5}$$

where

$$\widetilde{G}_N(y) = \Big(\sum_{z\in\mathbb{Z}^d} G^p(2Nz+y)\Big)^{1/p}$$

Applying Theorem 7.2 to the right hand side of (7.5) gives

$$\limsup_{m\to\infty}\frac{1}{m}\log\sum_{x_1,\cdots,x_m\in\mathbb{Z}^d}\bigg[\frac{1}{m!}\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m G(x_{\sigma(k)}-x_{\sigma(k-1)})\bigg]^p \le p\log\widetilde{\rho}_N$$

where

$$\widetilde{\rho}_N = \sup\bigg\{\sum_{x,y\in[-N,N)^d}\widetilde{G}_N(x-y)g(x)g(y);\quad \sum_{x\in[-N,N)^d}|g(x)|^{\frac{2p}{2p-1}}=1\bigg\}.$$

Finally, the upper bound for (7.2) follows from the fact that

$$\limsup_{N\to\infty}\widetilde{\rho}_N\le\rho.$$

The proof of (5.41) in Theorem 5.7 is more delicate. Indeed, the moment representation of I_∞ takes a form much more complicated than (7.1). To state it, we introduce the following notations.

For any $1\le l\le m$, let $\pi=(\pi_1,\cdots,\pi_l)$ represent a partition of $\{1,\cdots,m\}$, where $\pi_1,\cdots,\pi_l$ are disjoint, non-empty subsets of $\{1,\cdots,m\}$ satisfying $\{1,\cdots,m\}=\pi_1\cup\cdots\cup\pi_l$. Set

$$\mathcal{A}(\pi)=\Big\{(x_1,\cdots,x_m)\in\mathbb{Z}^d;\quad x_i=x_j \text{ for any } i,j\in\pi_k,\ k=1,\cdots l\Big\}.$$

For any $(x_1,\cdots,x_m)\in\mathcal{A}(\pi)$ and for any $1\le k\le l$, we use x_{π_k} for the common value of x_j $(j\in\pi_k)$. For $1\le l\le m$, write $\mathcal{E}_l$ for the set of the partitions π which divide $\{1,\cdots,m\}$ into l non-empty disjoint sets. By combinatorics

$$\#\{\mathcal{E}_l\}=\frac{1}{l!}\sum_{\substack{j_1+\cdots+j_l=m\\ j_1,\cdots,j_l\ge1}}\frac{m!}{j_1!\cdots j_l!}.$$

A computation comparable to (7.1) gives the following moment representation:

$$\mathbb{E}\,I_\infty^m=\sum_{x_1,\cdots,x_m\in\mathbb{Z}^d}\bigg[\sum_{l=1}^m\sum_{\pi\in\mathcal{E}_l}1_{\{(x_1,\cdots,x_m)\in\mathcal{A}(\pi)\}}\sum_{\sigma\in\Sigma_l}\prod_{k=1}^l G_1(x_{\pi_{\sigma(k)}}-x_{\pi_{\sigma(k-1)}})\bigg]^p \qquad (7.6)$$

where G_1 is defined in (5.45). The increase of complexity in (7.6) is caused by the difficulty in counting the time vectors $(i_1,\cdots,i_m)$ with non-distinct components.

By an argument harder than the one used for (7.2), the representation (7.6) leads to

$$\lim_{m\to\infty}\frac{1}{m}\log\frac{1}{(m!)^p}\mathbb{E}\,I_\infty^m=-p\log\gamma^{-1}(1). \tag{7.7}$$

Finally, the desired (5.41) follows from Lemma 3.1.

Another successful application of the method of high moment asymptotics is the proof of (3.8) in Theorem 3.2, which has been reduced to the proof of (3.16). Like the previous case, we can directly prove the lower bound

$$\liminf_{m\to\infty}\frac{1}{m}\log\int_{(\mathbb{R}^d)^m}d\lambda_1\cdots d\lambda_m\bigg[\frac{1}{m!}\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m Q\Big(\sum_{j=1}^k\lambda_{\sigma(j)}\Big)\bigg]^p\ge\log\rho_1. \tag{7.8}$$

As for the upper bound, we have the following discrete version.

Theorem 7.3. *Let $\pi(x)$ and $Q(x)$ be two non-negative functions on $\mathbb{Z}^d$ such that π is locally supported, $\pi(-x)=\pi(x)$ for all $x\in\mathbb{Z}^d$, and that $Q(x)\longrightarrow 0$ as $|x|\to\infty$. Then*

$$\limsup_{m\to\infty}\frac{1}{m}\log\sum_{x_1,\cdots,x_n\in\mathbb{Z}^d}\Big(\prod_{k=1}^m\pi(x_k)\Big)\Big[\frac{1}{m!}\sum_{\sigma\in\Sigma_m}\prod_{k=1}^m Q\Big(\sum_{j=1}^k x_{\sigma(j)}\Big)\Big]^p\le\log\widehat{\rho} \tag{7.9}$$

where

$$\widehat{\rho}=\sup_{|f|_2=1}\sum_{x\in\mathbb{Z}^d}\pi(x)\Big[\sum_{y\in\mathbb{Z}^d}\sqrt{Q(x+y)}\sqrt{Q(y)}f(x+y)f(y)\Big]^p \tag{7.10}$$

and

$$|f|_2=\Big(\sum_{x\in\mathbb{Z}^d}f^2(x)\Big)^{1/2}.$$

In order to apply Theorem 7.3 to the proof of the upper bound of (3.16), the major challenge is descretization. This has been done by the tool of Fourier analysis.

In summary, the method of high moment asymtotics is at its beginning stage and there are considerable challenges are to be expected. On the other hand, it opens up new avenues for the treatment of a wide range of problems arisen from area of sample path intersections, especially those out of the reach of the conventional approaches.

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