Central limit theorems for logarithmic averages

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We prove central limit theorems and related asymptotic results for

$$\int_1^T t^{-1} f\left(W(t)/\sqrt{t}\right) dt,\ T \to \infty \quad \text{and} \quad \sum_{k \le N} k^{-1} f\left(S_k/\sqrt{k}\right),\ \ N \to \infty$$

where W is a Wiener process and S_k are partial sums of i.i.d. random variables with mean 0 and variance 1. The integrability and smoothness conditions made on f are optimal in a number of important cases.

AMS 1999 subject classification: Primary 60F15, Secondary 60F05.
Key words and phrases: Almost sure central limit theorem, logarithmic averages.

*Supported by the Hungarian National Foundation for Scientific Research, Grant T 29621

1 Introduction

Let $X_1, X_2, \ldots$ be i.i.d. random variables with $EX_1 = 0$, $EX^2 = 1$ and let $S_n = X_1 + \ldots + X_n$. In the past decade several papers investigated the asymptotic properties of sums

$$\sum_{k \le N} \frac{1}{k} f\left(\frac{S_k}{\sqrt{k}}\right) \tag{1.1}$$

for various classes of measurable functions $f : \mathbf{R} \to \mathbf{R}$. In particular, considerable effect has been devoted to finding the most general conditions under which the relation

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{k \le N} \frac{1}{k} f\left(\frac{S_k}{\sqrt{k}}\right) = \int_{-\infty}^{+\infty} f(x)\phi(x)dx \qquad \text{a.s. for all } x \tag{1.2}$$

holds, where $\phi(x) = (2\pi)^{-1/2}e^{-x^2/2}$ is the standard normal density. Brosamler [3] and Schatte [13] proved (1.2) (assuming also $E|X_1|^\nu < +\infty$ for some $\nu > 2$) for indicator functions f while Lacey and Philipp [12] obtained (1.2) for a large class of smooth bounded functions f. For results in the unbounded case see Schatte [14], Berkes, Csáki, and Horváth [1], Ibragimov and Lifshits [10]. In particular, Ibragimov and Lifshits [10] proved that (1.2) holds if $f : \mathbf{R} \to \mathbf{R}$ is a continuous function satisfying

$$\int_{-\infty}^{+\infty} |f(x)|\phi(x)dx < +\infty \tag{1.3}$$

and

$$f(|x|) \text{ is nondecreasing and } f(|x|)e^{-cx^2} \text{ is nonincreasing for some } c > 0. \tag{1.4}$$

They also showed that condition (1.3) itself is not sufficient for (1.2) even for continuous f (see [10], Example 3) and thus some regularity condition of the type (1.4) is needed.

The purpose of the present paper is to prove central limit theorems and related asymptotic results for sums (1.1) and to get, in a number of important situations, essentially optimal results. In the case of indicator functions f, CLT's for (1.1) were proved by Weigl [15] and Csörgő and Horváth [6]; these results were extended for large classes of bounded functions f by Horváth and Khoshnevisan [8], [9] and Berkes and Horváth [2]. In this paper our main interest is the case of unbounded f and we shall prove the following results:

Theorem 1.1 *Let $\{W(t), t \geq 0\}$ be a Wiener process and $f : \mathbf{R} \to \mathbf{R}$ a measurable function satisfying*

$$\int_{-\infty}^{+\infty} f^2(x)\phi(x)dx < +\infty \quad \textit{and} \quad \int_{-\infty}^{+\infty} f(x)\phi(x)dx = 0. \tag{1.5}$$

Then

$$\frac{1}{(\log T)^{1/2}} \int_1^T \frac{1}{t} f\left(\frac{W(t)}{\sqrt{t}}\right) dt \xrightarrow{\mathcal{D}} N(0, \sigma_f^2) \quad \textit{as} \quad T \to \infty \tag{1.6}$$

$$\limsup_{T\to\infty} \frac{1}{(2\log T \log\log\log T)^{1/2}} \int_1^T \frac{1}{t} f\left(\frac{W(t)}{\sqrt{t}}\right) dt = \sigma_f \qquad \textit{a.s.} \tag{1.7}$$

and

$$\liminf_{T\to\infty} \left(\frac{\log\log\log T}{\log T}\right)^{1/2} \max_{1\leq s\leq T} \left| \int_1^s \frac{1}{t} f\left(\frac{W(t)}{\sqrt{t}}\right) dt \right| = \frac{\pi}{\sqrt{8}} \sigma_f \qquad \textit{a.s.} \tag{1.8}$$

where

$$\sigma_f^2 = 4 \sum_{k=1}^{\infty} \frac{1}{k!k} \langle f, h_k \rangle^2 \leq 4 \int_{-\infty}^{+\infty} f^2(x)\phi(x)dx \tag{1.9}$$

and $(h_n, n \geq 0)$ are the Hermite polynomials defined by

$$h_0(x) = 1, \quad h_n(x) = e^{x^2/2}(-1)^n \frac{d^n}{dx^n}(e^{-x^2/2}), \quad n = 1, 2, \ldots$$

The inner product $\langle \cdot, \cdot \rangle$ is meant in the Hilbert space $\mathcal{L}^2(\mathbf{R}, \phi(x)dx)$.

It is worth comparing Theorem 1.1 with a result of Brosamler [3] stating that if f satisfies (1.3) then

$$\lim_{T\to\infty} \frac{1}{\log T} \int_1^T \frac{1}{t} f\left(\frac{W(t)}{\sqrt{t}}\right) dt = \int_{-\infty}^{+\infty} f(x)\phi(x)dx \qquad \text{a.s.} \tag{1.10}$$

This result is optimal in the sense that nothing beyond the existence of the integral on the right hand side of (1.10) is assumed. Theorem 1.1 gives a similarly optimal result for the CLT and LIL where nothing beyond the natural second moment condition (1.5) is assumed. By the well known connection between Brownian motion and the Ornstein-Uhlenbeck process, (1.6)–(1.8) are equivalent, respectively, to the limit laws (2.1)–(2.3) in Theorem 2.1 for the additive functionals of the Markov chain associated with the Ornstein-Uhlenbeck process in our situation. Hence condition (1.5) is best possible, since it requires nothing more than a finite energy for the function f.

Theorem 1.2 *Let $X_1, X_2, \dots$ be i.i.d random variables with $EX_1 = 0$, $EX_1^2 = 1$ and $E|X_1|^\nu < +\infty$ for some $\nu > 2$. Let $S_n = X_1 + \dots + X_n$ and let $f : \mathbf{R} \to \mathbf{R}$ be a measurable function satisfying (1.5) and*

$$\int_{-\infty}^{+\infty} \sup_{|t|\le h} |f(x+t) - f(x)| e^{-x^2/2} dx = O\left(\frac{1}{\left(\log \frac{1}{h}\right)^\alpha}\right) \qquad \text{for some } \alpha > 1. \tag{1.11}$$

Then

$$\frac{1}{(\log N)^{1/2}} \sum_{k\le N} \frac{1}{k} f\left(\frac{S_k}{\sqrt{k}}\right) \xrightarrow{\mathcal{D}} N(0, \sigma_f^2) \quad \text{as } N \to \infty \tag{1.12}$$

$$\limsup_{N\to\infty} \frac{1}{(2 \log N \log\log\log N)^{1/2}} \sum_{k\le N} \frac{1}{k} f\left(\frac{S_k}{\sqrt{k}}\right) = \sigma_f \qquad a.s. \tag{1.13}$$

and

$$\liminf_{N\to\infty} \left(\frac{\log\log\log N}{\log N}\right)^{\frac{1}{2}} \max_{1\le m\le N} \left| \sum_{k=1}^{m} \frac{1}{k} f\left(\frac{S_k}{\sqrt{k}}\right) \right| = \frac{\pi}{\sqrt{8}} \sigma_f \qquad a.s. \tag{1.14}$$

where σ_f is defined by (1.9).

The smoothness condition (1.11) in Theorem 1.2 is essentially sharp, as the following theorem shows:

Theorem 1.3 *Let $X_1, X_2, \dots$ be i.i.d. random variables with $P(X_1 = 1) = P(X_1 = -1) = 1/2$ and let $S_n = X_1 + \dots + X_n$. There exists a continuous function $f : \mathbf{R} \to \mathbf{R}$ satisfying (1.5) such that*

$$\int_{-\infty}^{+\infty} \sup_{|t|\le h} |f(x+t) - f(x)| e^{-cx^2} dx = O\left(\frac{1}{\left(\log \frac{1}{h}\right)^\alpha}\right) \tag{1.15}$$

for some $c > 0$, $\alpha > 0$ and

$$\limsup_{N\to\infty} \frac{1}{\log N} \sum_{k\le N} \frac{1}{k} f\left(\frac{S_k}{\sqrt{k}}\right) = +\infty \quad a.s. \tag{1.16}$$

and consequently (1.12)–(1.14) are false.

The moment condition $E|X_1|^\nu < +\infty$, $\nu > 2$ in Theorem 1.2 can be weakened at the cost of strengthening the smoothness condition (1.11). In fact, we have

Theorem 1.4 *Let $X_1, X_2, \ldots$ be i.i.d. random variables with $EX_1 = 0$, $EX_1^2 = 1$ and $EH(|X_1|) < +\infty$ where H is a continuous, nonnegative function on $[0, \infty)$ such that $H(t)/(t^2 \log\log t)$ is nondecreasing and $H(t)/t^3$ in nonincreasing. Let $S_n = X_1 + \ldots + X_n$ and let $f : \mathbf{R} \to \mathbf{R}$ be a measurable function satisfying (1.5) and*

$$\int_{-\infty}^{+\infty} \sup_{|t| \le h} |f(x+t) - f(x)| e^{-x^2/2} dx = O(\psi(h))$$

where

$$\int_1^\infty \frac{1}{t} \psi\left(\frac{H^{-1}(t)}{\sqrt{t}}\right) dt < +\infty.$$

Then relations (1.12), (1.13), and (1.14) hold.

For example, if $X_1, X_2, \ldots$ are i.i.d. random variables with $EX_1 = 0$, $EX_1^2 = 1$ and $EX_1^2 (\log^+ |X_1|)^\beta < +\infty$ for some $\beta > 2$ then relations (1.12)–(1.14) hold if (1.5) is valid and

$$\int_{-\infty}^{+\infty} \sup_{|t| \le h} |f(x+t) - f(x)| e^{-x^2/2} dx = O(h^\gamma) \quad \text{for some } \gamma > 2/\beta.$$

In conclusion we note that the method of the proofs of Theorems 1.1–1.4 provides new information also on the law of large numbers (1.2). In fact, we have

Theorem 1.5 *Let $X_1, X_2, \ldots$ be i.i.d. random variables with $EX_1 = 0$, $EX_1^2 = 1$ and $E|X_1|^\nu < +\infty$ for some $\nu > 2$. Let $S_n = X_1 + \cdots + X_n$ and let $f : \mathbf{R} \to \mathbf{R}$ be a measurable function satisfying (1.3) and (1.11). Then (1.2) holds. If instead of (1.11) we assume only that (1.15) holds for some $c > 0$, $\alpha > 0$, then (1.2) is generally false.*

Thus, under the additional moment condition $E|X_1|^\nu < +\infty$, $\nu > 2$, the monotonicity condition (1.4) of Ibragimov and Lifshits can be replaced by the smoothness condition (1.11) and this condition is essentially sharp.

2 Proofs

The proof of Theorem 1.1 is based upon the following result for functionals of positive recurrent Markov chains.

Theorem 2.1 *Let $\{X_n\}_{n \ge 0}$ be a positive recurrent Markov chain with the state space $(E, \mathcal{E})$ (where $\mathcal{E}$ is countably generated), transition probability $P(x, A)$ and invariant distribution π. Let $f : E \to \mathbf{R}$ be a measurable function satisfying* *(i)* $\int f(x)\pi(dx) = 0$ *and* $\int f^2(x)\pi(dx) < +\infty$

(ii) $\sum_{k=1}^{\infty} f(\cdot)P^k f(\cdot)$ *converges in* $\mathcal{L}(E, \mathcal{E}, \pi)$.

Then

$$n^{-1/2}\sum_{k=1}^{n} f(X_k) \xrightarrow{\mathcal{D}} N(0, \sigma_f^2) \tag{2.1}$$

$$\limsup_{n\to\infty} (2n\log\log n)^{-1/2}\sum_{k=1}^{n} f(X_k) = \sigma_f \quad a.s. \tag{2.2}$$

and

$$\liminf_{n\to\infty} \sqrt{\frac{\log\log n}{n}} \max_{k\le n}\Big|\sum_{j=1}^{k} f(X_j)\Big| = \frac{\pi}{\sqrt{8}}\sigma_f \quad a.s. \tag{2.3}$$

where

$$\sigma_f^2 = \int f^2(x)\pi(dx) + 2\sum_{k=1}^{\infty}\int f(x)P^k f(x)\pi(dx). \tag{2.4}$$

Proof. (2.1) and (2.2) are given in Corollary 1.4 of Chen [4]; (2.3) is given in Theorem 1.1 of Chen [5].

We now consider a 1-dimensional Ornstein-Uhlenbeck process X_t with the infinitesimal generator

$$Af(x) = \frac{1}{2}f''(x) - \frac{x}{2}f'(x).$$

It is well known that

(a) X_t is a positive recurrent Markov process with invariant distribution $\phi(x)dx$

(b) X_t has the representation $X_t = e^{-t/2}W(e^t)$.

Under the variable substitution $s \longrightarrow \log s$, therefore, the limit laws stated in Theorem 1.1 are equivalent, respectively, to

$$\frac{1}{\sqrt{t}}\int_0^t f(X_s)ds \xrightarrow{\mathcal{D}} N(0, \sigma_f^2)$$

$$\limsup_{t\to\infty} \frac{1}{\sqrt{2t\log\log t}}\int_0^t f(X_s)ds = \sigma_f \quad a.s.$$

$$\liminf_{t\to\infty}\sqrt{\frac{\log\log t}{t}}\max_{0\le s\le t}\Big|\int_0^s f(X_u)du\Big| = \frac{\pi}{\sqrt{8}}\sigma_f \quad a.s.$$

We prove these laws by skeleton approximation. Namely, we apply Theorem 2.1 to the Markov chain

$$\widetilde{X}_n \equiv \Big(X_n, \int_{n-1}^{n} f(X_s)ds\Big).$$

One can verify that this chain is positive recurrent with its k-step transition $\widetilde{P}^k$ ($k \ge 1$) given by

$$\widetilde{P}^k((x,y),\cdot) = P_x\Big\{\Big(X_k, \int_{k-1}^{k} f(X_s)ds\Big) \in \cdot\Big\} \quad (x,y)\in \mathbf{R}^2$$

and invariant distribution $\widetilde{\pi}$ defined by

$$\widetilde{\pi}(\cdot) = \int_{-\infty}^{\infty} P_x\Big\{\Big(X_1, \int_0^1 f(X_s)ds\Big) \in \cdot\Big\}\phi(x)dx.$$

Let $\tilde{f}$: $\mathbf{R}^2 \longrightarrow \mathbf{R}$ be the canonical projection given by $\tilde{f}(x,y) = y$. We prove that $\tilde{f}$ satisfies conditions (i) and (ii) in Theorem 2.1. Indeed,

$$\int \tilde{f}(x,y)\widetilde{\pi}(d(x,y)) = \int_{-\infty}^{\infty} E_x\Big[\int_0^1 f(X_s)ds\Big]\phi(x)dx = 0$$

and

$$\begin{aligned}\int \tilde{f}^2(x,y)\widetilde{\pi}(d(x,y)) &= \int_{-\infty}^{\infty} E_x\Big[\int_0^1 f(X_s)ds\Big]^2\phi(x)dx \\ &\le \int_{-\infty}^{\infty} f^2(x)\phi(x)dx < \infty.\end{aligned} \tag{2.5}$$

Hence we have (i). To show (ii), it is enough to prove

$$\sum_{k=1}^{\infty}\int |\tilde{f}^2(x,y)|\cdot|\widetilde{P}^k\tilde{f}(x,y)|\widetilde{\pi}(d(x,y)) < \infty.$$

Note that

$$\widetilde{P}^k\tilde{f}(x,y) = \int_{k-1}^{k} P_s f(x)ds$$

further

$$\int |\tilde{f}(x,y)|\cdot|\widetilde{P}^k\tilde{f}(x,y)|\widetilde{\pi}(d(x,y))$$

$$= \int_{-\infty}^{\infty} E_x\Big(\Big|\int_0^1 f(X_s)ds\Big|\cdot\Big|\int_{k-1}^k P_tf(X_1)dt\Big|\Big)\phi(x)dx$$

(2.6)

$$\le \int_0^1\int_{k-1}^k\Big[\int_{-\infty}^{\infty}|f(x)|\cdot E_x|P_tf(X_{1-s})|\phi(x)dx\Big]dtds$$

$$\le ||f||_2\int_{k-1}^k\Big(\int_{-\infty}^{\infty}|P_tf(x)|^2\phi(x)dx\Big)^{1/2}dt$$

and

$$\int \tilde{f}(x,y)\widetilde{P}^k\tilde{f}(x,y)\widetilde{\pi}(d(x,y))$$

(2.7)

$$= \int_0^1\int_{k-1}^k\Big[\int_{-\infty}^{\infty}f(x)P_{t+1-s}f(x)\phi(x)dx\Big]dtds$$

where $\{P_s\}$ $(s\ge 0)$ is the transition semigroup of the Ornstein-Uhlenbeck process X_t. On the other hand, it is known that

$$Ah_n = -\frac{n}{2}h_n \qquad n=1,2,\ldots$$

and

$$\int_{-\infty}^{+\infty} h_m(x)h_n(x)\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx = \begin{cases} n! & m=n \\ 0 & m\ne n. \end{cases}$$

Let $e_n=(n!)^{-1/2}h_n$. Then $\{e_n\}_{n\ge 0}$ is a standard orthogonal basis of $\mathcal{L}^2(\mathbf{R},\phi(x)dx)$. Since f is orthogonal to e_0,

$$f(x)=\sum_{k=1}^{\infty}\langle f,e_k\rangle\, e_k(x)$$

and

$$P_tf(x)=\sum_{k=1}^{\infty}\langle f,e_k\rangle\, P_te_k(x)$$

(2.8)

$$=\sum_{k=1}^{\infty}\langle f,e_k\rangle\, e^{tA}e_k(x)=\sum_{k=1}^{\infty}\langle f,e_k\rangle\exp\{-2^{-1}kt\}e_k(x).$$

By the Cauchy-Schwarz inequality

$$
\begin{aligned}
|P_t f(x)| &\leq \Big(\sum_{k=1}^{\infty} \langle f, e_k\rangle^2\Big)^{1/2} \Big(\sum_{k=1}^{\infty} \exp\{-kt\} e_k^2(x)\Big)^{1/2} \\
&= ||f||_2 \Big(\sum_{k=1}^{\infty} \exp\{-kt\} e_k^2(x)\Big)^{1/2}.
\end{aligned}
$$

Hence by (2.6),

$$
\begin{aligned}
&\int |\tilde{f}(x,y)| \cdot |\widetilde{P}^k \tilde{f}(x,y)| \widetilde{\pi}(d(x,y)) \\
&\leq ||f||_2^2 \int_{k-1}^{k} \Big(\sum_{k=1}^{\infty} \exp\{-kt\}\Big)^{1/2} dt \\
&= ||f||_2^2 \int_{k-1}^{k} \frac{\exp\{-2^{-1}t\}}{(1-\exp\{-t\})^{1/2}} dt.
\end{aligned}
$$

Consequently

$$
\begin{aligned}
&\sum_{k=1}^{\infty} \int |\tilde{f}(x,y)| \cdot |\widetilde{P}^k \tilde{f}(x,y)| \widetilde{\pi}(d(x,y)) \\
&\leq ||f||_2^2 \int_0^{\infty} \frac{\exp\{-2^{-1}t\}}{(1-\exp\{-t\})^{1/2}} dt = \pi ||f||_2^2 < +\infty
\end{aligned}
$$

which gives (ii). Therefore, Theorem 2.1 applies to the Markov chain $\{\widetilde{X}_n\}_{n\geq 0}$ and function $\tilde{f}$ that gives

$$
\frac{1}{\sqrt{n}} \int_0^n f(X_s) ds \xrightarrow{\mathcal{D}} N(0, \sigma_f^2)
$$

$$
\limsup_{n\to\infty} \frac{1}{\sqrt{2n \log\log n}} \int_0^n f(X_s) ds = \sigma_f \quad a.s.
$$

$$
\liminf_{n\to\infty} \sqrt{\frac{\log\log n}{n}} \max_{0\leq s\leq n} \Big| \int_0^s f(X_u) du \Big| = \frac{\pi}{\sqrt{8}} \sigma_f \quad a.s.
$$

It remains to identify σ_f^2. By (2.4), (2.5), (2.7), (2.8) we have

$$\begin{aligned}
\sigma_f^2 &= \int \tilde{f}^2(x,y)\widetilde{\pi}(d(x,y)) + 2\sum_{k=1}^{n}\int \tilde{f}(x,y)\widetilde{P}^k\tilde{f}(x,y)\widetilde{\pi}(d(x,y)) \\
&= \int_{-\infty}^{\infty} E_x\Big[\int_0^1 f(X_s)ds\Big]^2\phi(x)dx + 2\int_0^1\int_0^{\infty}\Big[\int_{-\infty}^{\infty} f(x)P_{t+1-s}f(x)\phi(x)dx\Big]dtds \\
&= 2\iint_{0\le s\le t\le 1}\Big[\int_{-\infty}^{\infty} f(x)P_{t-s}f(x)\phi(x)dx\Big]dtds \\
&\qquad\qquad + 2\int_0^1\int_{1-s}^{\infty}\Big[\int_{-\infty}^{\infty} f(x)P_t f(x)\phi(x)dx\Big]dtds \\
&= 2\int_0^{\infty}\Big[\int_{-\infty}^{\infty} f(x)P_t f(x)\phi(x)dx\Big]dt = 4\sum_{k=1}^{\infty}\frac{1}{k}\langle f,e_k\rangle^2 \\
&= 4\sum_{k=1}^{\infty}\frac{1}{kk!}\langle f,h_k\rangle^2 \le 4\int_{-\infty}^{\infty} f^2(x)\phi(x)dx
\end{aligned}$$

and Theorem 1.1 is proved.

Theorem 1.2 can be easily deduced from Theorem 1.1 by using the Komlós-Major-Tusnády approximation theorem (see [11]). Indeed, $EX_1 = 0$, $EX_1^2 = 1$ and $E|X_1|^\nu < +\infty$, $\nu > 2$ imply that there exists a Wiener process $\{W(t), t\ge 0\}$ such that setting $S(t) = \sum_{1\le k\le t} X_k$ we have

$$|S(t) - W(t)| = o(t^{1/\nu}) \quad \text{a.s. as } t\to\infty. \tag{2.9}$$

Noting that $|[t]^{-1/2} - t^{-1/2}| \le 2t^{-3/2}$ for $t\ge 2$ and $|S(t)|/t^{3/2} = O(1)$ a.s., relation (2.9) implies

$$\left|\frac{S(t)}{[t]^{1/2}} - \frac{W(t)}{t^{1/2}}\right| = o(t^{1/\nu-1/2}) \quad \text{a.s. as } t\to\infty$$

and thus there exists a $T = T(\omega)$ such that

$$\left|\frac{S(t)}{[t]^{1/2}} - \frac{W(t)}{t^{1/2}}\right| \le t^{1/\nu-1/2} \quad \text{if } t\ge T. \tag{2.10}$$

We show that

$$I := \int_1^{\infty}\frac{1}{t}\left|f\left(\frac{S(t)}{[t]^{1/2}}\right) - f\left(\frac{W(t)}{t^{1/2}}\right)\right|dt < +\infty \quad \text{a.s.} \tag{2.11}$$

Indeed, $I \le I_1 + I_2$ where

$$I_1 = \int_1^T \frac{1}{t} \left| f\left(\frac{S(t)}{[t]^{1/2}}\right) - f\left(\frac{W(t)}{t^{1/2}}\right) \right| dt$$

and

$$\begin{aligned} I_2 &= \int_T^\infty \frac{1}{t} \left| f\left(\frac{S(t)}{[t]^{1/2}}\right) - f\left(\frac{W(t)}{t^{1/2}}\right) \right| dt \\ &\le C \int_T^\infty \frac{1}{t} \sup_{|h| \le t^{1/\nu - 1/2}} \left| f\left(\frac{W(t)}{t^{1/2}} + h\right) - f\left(\frac{W(t)}{t^{1/2}}\right) \right| dt \end{aligned}$$

by using (2.10). Thus by (1.11) we get

$$\begin{aligned} EI_2 &\le C' \int_1^\infty \frac{1}{t} E \sup_{|h| \le t^{1/\nu - 1/2}} \left| f\left(\frac{W(t)}{t^{1/2}} + h\right) - f\left(\frac{W(t)}{t^{1/2}}\right) \right| dt \\ &= C' \int_1^\infty \frac{dt}{t} \int_{-\infty}^{+\infty} \sup_{|h| \le t^{1/\nu - 1/2}} |f(x+h) - f(x)| \phi(x) dx \\ &\le C'' \int_1^\infty \frac{1}{t} (\log t^{1/2 - 1/\nu})^{-\alpha} dt < +\infty \end{aligned}$$

using $\alpha > 1$. Hence we see that $I_2 < +\infty$ a.s. On the other hand, $f(S(t)/[t]^{1/2})$ is constant over each interval $[k, k+1)$ and thus $t^{-1}|f(S(t)/[t]^{1/2})|$ is integrable over any finite subinterval of $[1, +\infty)$. The same holds for $t^{-1}|f(W(t)/t^{1/2})|$, since

$$E \int_1^L \frac{1}{t} \left| f\left(\frac{W(t)}{t^{1/2}}\right) \right| dt \le \int_1^L \frac{dt}{t} \int_{-\infty}^{+\infty} |f(x)| \phi(x) dx < +\infty \tag{2.12}$$

for any $1 \le L < +\infty$ by (1.5). Thus we see that $I_1 < +\infty$ a.s. and (2.11) is proved. As a consequence we have

$$\int_1^\infty \frac{1}{t^2} \left| f\left(\frac{S(t)}{[t]^{1/2}}\right) - f\left(\frac{W(t)}{t^{1/2}}\right) \right| dt < +\infty \quad \text{a.s.} \tag{2.13}$$

and a computation similar to (2.12) shows that

$$\int_1^\infty \frac{1}{t^2} \left| f\left(\frac{W(t)}{t^{1/2}}\right) \right| dt < +\infty \quad \text{a.s.} \tag{2.14}$$

which, together with (2.13), gives

$$\int_1^\infty \frac{1}{t^2}\left|f\left(\frac{S(t)}{[t]^{1/2}}\right)\right| < +\infty \quad \text{a.s.} \tag{2.15}$$

Thus for any $N \geq 1$ we get

$$\begin{aligned}
&\left|\sum_{k\leq N}\frac{1}{k}f\left(\frac{S_k}{k^{1/2}}\right) - \int_1^N \frac{1}{t}f\left(\frac{W(t)}{t^{1/2}}\right)dt\right| \\
&= \left|\int_1^N \frac{1}{[t]}f\left(\frac{S(t)}{[t]^{1/2}}\right)dt - \int_1^N \frac{1}{t}f\left(\frac{W(t)}{t^{1/2}}\right)dt\right| \\
&\leq \int_1^N \left(\frac{1}{[t]}-\frac{1}{t}\right)\left|f\left(\frac{S(t)}{[t]^{1/2}}\right)\right|dt + \int_1^N \frac{1}{t}\left|f\left(\frac{S(t)}{[t]^{1/2}}\right) - f\left(\frac{W(t)}{t^{1/2}}\right)\right|dt \\
&\leq \int_1^\infty \frac{1}{t^2}\left|f\left(\frac{S(t)}{[t]^{1/2}}\right)\right|dt + \int_1^\infty \frac{1}{t}\left|f\left(\frac{S(t)}{[t]^{1/2}}\right) - f\left(\frac{W(t)}{t^{1/2}}\right)\right|dt \\
&= O(1),
\end{aligned}$$

since the last two integrals are finite by (2.15) and (2.11). Hence Theorem 1.2 follows from Theorem 1.1. The proof of Theorem 1.4 is essentially the same, just instead of the a.s. approximation (2.9), valid under $EX_1 = 0$, $EX_1^2 = 1$, $E|X_1|^\nu < +\infty$ $(\nu > 2)$, we use the approximation

$$|S(t) - W(t)| = O(H^{-1}(t)) \tag{2.16}$$

valid under $EX_1 = 0$, $EX_1^2 = 1$, $EH(|X_1|) < +\infty$ (provided the regularity conditions made on H hold). For a proof of the general a.s. invariance principle (2.16) we refer to Einmahl [7].

We finally prove Theorem 1.3. In the construction that follows, we utilize the basic idea of Example 3 of Ibragimov and Lifshits [10], but the details are considerably more complicated. Let $\psi_{a,h}(x)$ denote the function in $(-\infty, +\infty)$ which equals 1 for $x = a$, equals 0 for $x \leq a-h$ and $x \geq a+h$ and is linear in the intervals $[a-h, a]$ and $[a, a+h]$. Let $\delta_k = (2^{2^{4^k}})^{-6}$ and define the function f by

$$f(x) = \sum_{k=1}^\infty \sum_{\substack{(i,j):\, i\geq 0,\ 1\leq j\leq 2^{2^{4^k}} \\ 2^{k-3}\leq\sqrt{i/j}\leq 2^{k+1}}} (2^{4^k})^2 \psi_{\sqrt{i/j},\delta_k}(x). \tag{2.17}$$

Let $X_2, X_2, \ldots$ be i.i.d. random variables with $P(X_1 = 1) = P(X_1 = -1) = 1/2$. We show that

$$\int_{-\infty}^{+\infty} f^2(x)\phi(x)dx < +\infty \tag{2.18}$$

$$\int_{-\infty}^{+\infty} \sup_{|t|\le h} |f(x+t) - f(x)| e^{-cx^2} dx = O\left(\frac{1}{\left(\log \frac{1}{h}\right)^{\alpha}}\right) \tag{2.19}$$

for some $c > 0, \alpha > 0$ and

$$\limsup_{N\to\infty} \frac{1}{\log N} \sum_{k\le N} \frac{1}{k} f\left(\frac{S_k}{\sqrt{k}}\right) = +\infty \quad \text{a.s.} \tag{2.20}$$

Letting $\mu = \int_{-\infty}^{+\infty} f(x)\phi(x)dx$ and replacing f by $f - \mu$, we get a function satisfying all requirements of Theorem 1.3.

To prove (2.18) let

$$H_k = \{\sqrt{i/j} : i \ge 0,\ 1 \le j \le 2^{2^{4^k}}\} \cap [2^{k-3}, 2^{k+1}].$$

Observe that if two rational numbers i/j and i'/j' satisfy $i, i' \ge 0$, $1 \le j, j' \le 2^{2^{4^k}}$, then either $i/j = i'/j'$ or

$$|i/j - i'/j'| \ge \frac{1}{jj'} \ge \frac{1}{(2^{2^{4^k}})^2}.$$

This implies that any two different elements of the set H_k have distance

$$|\sqrt{i/j} - \sqrt{i'/j'}| = \frac{|i/j - i'/j'|}{\sqrt{i/j} + \sqrt{i'/j'}} \ge \frac{1}{(2^{2^{4^k}})^2 2^{k+2}}$$

and thus the functions $\psi_{\sqrt{i/j},\delta_k}(x)$ in the inner sum of (2.17) have disjoint supports. It also follows that the cardinality $|H_k|$ of the set H_k is at most $(2^{2^{4^k}})^2 2^{2k+3}$. Since

$$\int_{-\infty}^{+\infty} \psi^2_{\sqrt{i/j},\delta_k}(x)dx \le 2(2^{2^{4^k}})^{-6}$$

we get

$$\left(\int_{-\infty}^{+\infty} f^2(x)dx\right)^{1/2} \le \sum_{k=1}^{\infty} |H_k| (2^{4^k})^2 \sqrt{2} (2^{2^{4^k}})^{-3}$$
$$\le \sqrt{2} \sum_{k=1}^{\infty} (2^{2^{4^k}})^{-1} (2^{4^k})^2 2^{2k+3} \le \text{ const. } \sum_{k=1}^{\infty} (2^{2^{4^k}})^{-1/2} < +\infty,$$

proving (2.18).

To prove (2.19) if suffices to show that for any $g(x) = (2^{4^k})^2 \psi_{\sqrt{i/j},\delta_k}(x)$ in the inner sum in (2.17) we have

$$\sup_{|t|\le h} |g(x+t) - g(x)| \le \frac{c_1}{\left(\log \frac{1}{h}\right)^{\alpha}} e^{c_2 4^{k-3}} \tag{2.21}$$

for all real x and all $0 < h < 1$ where $c_1 > 0$, $0 < c_2 < 90$ and $0 < \alpha \le 1$ are absolute constants. Indeed, the support of such a $g(x)$ is $\subset [2^{k-3}, 2^{k+1}]$ and thus the factor $e^{c_2 4^{k-3}}$ in (2.21) is $\le e^{c_2 x^2}$ in this interval. Hence (2.21) implies

$$\int_{-\infty}^{+\infty} \sup_{|t|\le h} |g(x+t) - g(x)| e^{-90x^2} dx \le \frac{c_3}{(\log \frac{1}{h})^{\alpha}}. \tag{2.22}$$

Now the functions $\psi_{\sqrt{i/j},\delta_k}(x)$ in the inner sum of (2.17) have disjoint supports and thus (2.22) will hold also for the function

$$g = g_k = \sum_{\substack{(i,j):\, i\ge 0,\ 1\le j\le 2^{2^{4^k}} \\ 2^{k-3}\le\sqrt{i/j}\le 2^{k+1}}} (2^{4^k})^2 \psi_{\sqrt{i/j},\delta_k}(x).$$

Finally, the supports of g_k and g_ℓ are disjoint if $|k-\ell| > 4$ and thus (2.22) will be valid also for $g = \sum_{k=1}^{\infty} g_k$.

To prove (2.21) it suffices, in view of the graph of g, to verify that

$$|g(x+h) - g(x)| \le \frac{c_1}{\left(\log \frac{1}{h}\right)^{\alpha}} e^{c_2 4^{k-3}} \tag{2.23}$$

for

$$\sqrt{i/j} \le x < x+h \le \sqrt{i/j} + \delta_k.$$

Replacing h by λh for some $0 < \lambda < 1$, the left hand side of (2.23) will be multiplied by λ and the right hand side will be multiplied by

$$\left(\frac{\log \frac{1}{h}}{\log \frac{1}{\lambda h}}\right)^{\alpha} \ge \lambda^{\alpha} \ge \lambda.$$

Thus is suffices to verify (2.23) for the maximal value of h, i.e. for $x = \sqrt{i/j}$, $h = \delta_k$, when

$$|g(x+h) - g(x)| = (2^{4^k})^2, \qquad \log \frac{1}{h} = 6 \log 2 \cdot 2^{4^k}$$

and thus (2.23) holds if $0 < \alpha < 1$ is small enough and $c_2 > 128 \log 2 \approx 88.7\ldots$

Finally, to prove (2.20) it suffices to show that

$$P\left\{ \sum_{j=2^{n-1}+1}^{2^n} \frac{1}{j} f\left(\frac{S_j}{\sqrt{j}}\right) \geq \frac{1}{2} n^2 \ \text{ i.o.} \right\} = 1. \tag{2.24}$$

In order to establish (2.24), we note that Strassen's LIL implies that a.s. for infinitely many n we have

$$\frac{1}{2}\sqrt{\log n} \leq \frac{S_j}{\sqrt{j}} \leq 2\sqrt{\log n} \quad \text{for all } \ 2^{n-1} < j \leq 2^n. \tag{2.25}$$

Choose such an n and let $k \geq 1$ be defined by $2^{4^{k-1}} \leq n < 2^{4^k}$. Then (note that $\log 2 \approx 0.69\ldots$)

$$2^{k-2} \leq \sqrt{\log n} \leq 2^k$$

and thus by (2.25)

$$2^{k-3} \leq \frac{S_j}{\sqrt{j}} \leq 2^{k+1} \quad \text{for all} \quad 2^{n-1} < j \leq 2^n. \tag{2.26}$$

Also, in (2.26) we have $S_j/\sqrt{j} = \sqrt{S_j^2/j} = \sqrt{i/j}$ for some integers $i \geq 0$ and $1 \leq j \leq 2^{2^{4^k}}$. Moreover, (2.26) shows that $2^{k-3} \leq \sqrt{i/j} \leq 2^{k+1}$ and thus $S_j/\sqrt{j}$ belongs to the set H_k. Hence by the definition of f we have

$$f\left(\frac{S_j}{\sqrt{j}}\right) \geq (2^{4^k})^2 \geq n^2 \quad \text{for all} \quad 2^{n-1} < j \leq 2^n.$$

Consequently,

$$\sum_{j=2^{n-1}+1}^{2^n} \frac{1}{j} f\left(\frac{S_j}{\sqrt{j}}\right) \geq n^2 \sum_{j=2^{n-1}+1}^{2^n} \frac{1}{j} \geq \frac{1}{2} n^2$$

and (2.24) is proved.

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